fractal and fractional

MDPI

Article

On New Generalized Hermite–Hadamard–Mercer-Type Inequalities for Raina Functions

Zeynep Çiftci [1], Merve Coşkun [1], Çetin Yildiz [1,*], Luminiţa-Ioana Cotîrlă [2] and Daniel Breaz [3]

[1] Department of Mathematics, K.K. Education Faculty, Atatürk University, Erzurum 25240, Turkey; zbayrakdar@atauni.edu.tr (Z.Ç.); merve.coskun17@ogr.atauni.edu.tr (M.C.)
[2] Department of Mathematics, Technical University of Cluj-Napoca, 400020 Cluj-Napoca, Romania; luminita.cotirla@math.utcluj.ro
[3] Department of Mathematics, Decembrie University of Alba-Iulia, 510009 Alba-Iulia, Romania; dbreaz@uab.ro
* Correspondence: cetin@atauni.edu.tr

Abstract: In this research, we demonstrate novel Hermite–Hadamard–Mercer fractional integral inequalities using a wide class of fractional integral operators (the Raina fractional operator). Moreover, a new lemma of this type is proved, and new identities are obtained using the definition of convex function. In addition to a detailed derivation of a few special situations, certain known findings are summarized. We also point out that some results in this study, in some special cases, such as setting $\alpha = 0 = \varphi, \gamma = 1$, and $w = 0, \sigma(0) = 1, \lambda = 1$, are more reasonable than those obtained. Finally, it is believed that the technique presented in this paper will encourage additional study in this field.

Keywords: Hermite–Hadamard inequality; Jensen–Mercer inequality; Hölder inequality; Raina fractional operator

MSC: 26D15; 26D10; 26A51; 26A33; 33B20

check for updates

Citation: Çiftci, Z.; Coşkun, M.; Yildiz, Ç.; Cotîrlă, L.-I.; Breaz, D. On New Generalized Hermite–Hadamard–Mercer-Type Inequalities for Raina Functions. *Fractal Fract.* **2024**, *8*, 472. https://doi.org/10.3390/fractalfract8080472

Academic Editor(s): Seth Kermausuor and Eze Nwaeze

Received: 27 June 2024
Revised: 30 July 2024
Accepted: 2 August 2024
Published: 13 August 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction and Preliminaries

Two basic concepts in mathematics are the classical convexity and concavity of functions. Convex functions, in particular, differ from other function classes in that their definition has a geometric interpretation and they have numerous applications in all areas of mathematics (especially the theory of inequality), optimization theory, statistics, and applied sciences. Moreover, they are a cornerstone of inequality theory and have developed into the main motivating factor behind a number of inequalities. Convex functions have several applications (such as fractional operators, different convexities, and coordinates) in statistical and mathematical analysis, but the most important one has been demonstrated by the inequality theory [1–3]. In this regard, a number of traditional and analytical inequalities, particularly Hermite–Hadamard-, Fejér-, Hermite–Hadamard–Mercer-, Ostrowski-, Ostrowski–Mercer-, Chebyshev-, Opial-, Hardy-, Simpson-, and Jensen-type inequalities, have been proven [4–6]. Also, mathematicians and scientists can see them in research articles, monograph,s and textbooks devoted to the theory of convex analysis [7,8].

One of the fundamental functions of the theory of inequalities is the convex function. This is defined as follows:

Definition 1. *The mapping $f : \mathcal{I} \to \mathbb{R}$, is said to be a convex function if the following inequality holds:*

$$f(\kappa\varpi_1 + (1-\kappa)\varpi_2) \leq \kappa f(\varpi_1) + (1-\kappa)f(\varpi_2)$$

for all $\varpi_1, \varpi_2 \in \mathcal{I}$, and $\kappa \in [0,1]$. If $(-f)$ is convex, then f is said to be concave.

The subject of convex functions has been a focus of attention in recent decades, with the original notion extended and generalized in various directions. These functions play an

important role in many areas of analysis and geometry, and their properties have been the subject of detailed study. Readers interested in the aforementioned developments may wish to consult [9–11], which presents a comprehensive overview of these branches.

The Hermite–Hadamard inequality represents the most extensively researched inequality pertaining to the convexity attribute of functions. This inequality provides a necessary and sufficient condition for a function to be convex. Over the past few years, the theory of inequality has significantly advanced. The Hermite–Hadamard inequality, which has been a major driving factor behind this development, is one example of an important inequality that has emerged to demonstrate this. Convexity and inequality theories are strongly connected to one another, which is an essential point to consider. Novel convexity has seen a number of new definitions, extensions, and generalizations in recent years. Related developments in the theory of convexity inequality, especially integral inequalities theory, have also received significant attention. Formally, the Hermite–Hadamard inequality is defined as follows:

Let $f : I = [\varpi_1, \varpi_2] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on closed interval $[\varpi_1, \varpi_2]$, meaning that

$$f\left(\frac{\varpi_1 + \varpi_2}{2}\right) \leq \frac{1}{\varpi_2 - \varpi_1} \int_{\varpi_1}^{\varpi_2} f(\kappa) d\kappa \leq \frac{f(\varpi_1) + f(\varpi_2)}{2} \tag{1}$$

If f is a concave function, then the inequality in (1) will hold in the reverse direction.

Based on geometry, the Hermite–Hadamard inequality provides an upper and lower boundary for the integral mean of any convex function defined in a closed, bounded domain that contains the function's domain's ends and midpoint.

There are several aesthetic inequalities for convex functions in the literature, and Jensen's inequality has a special place among them. Jensen's inequality is presented as follows:

Let $0 \leq \varkappa_1 \leq \varkappa_2 \leq \ldots \leq \varkappa_n$, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be non-negative weights such that $\sum_{i=1}^{n} \mu_i = 1$. If f is a convex function on an interval containing $\varkappa_i$, for all $i \in \{1, 2 \cdots, n\}$, then

$$f\left(\sum_{i=1}^{n} \mu_i \varkappa_i\right) \leq \sum_{i=1}^{n} \mu_i f(\varkappa_i). \tag{2}$$

The following inequality is known as the Jenser–Mercer inequality in [12]:

$$f\left(\varpi_1 + \varpi_2 - \sum_{i=1}^{n} \mu_i \varkappa_i\right) \leq f(\varpi_1) + f(\varpi_2) - \sum_{i=1}^{n} \mu_i f(\varkappa_i), \tag{3}$$

$\forall \varkappa_i \in [\varpi_1, \varpi_2]$ and $\mu_i \in [0, 1]$, where f is a convex function on $[\varpi_1, \varpi_2]$.

Pavić, in [13], presented a generalized version of the Jensen–Mercer inequality as follows:

Assume that $f : [\varpi_1, \varpi_2] \to \mathbb{R}$ is a convex function and $\varkappa_i \in [\varpi_1, \varpi_2]$ are n-points. Let $\alpha, \varphi, \mu_i \in [0, 1]$, $\gamma \in [-1, 1]$ be coefficients of sums $\alpha + \varphi + \gamma = \sum_{i=1}^{n} \mu_i = 1$, meaning that

$$f\left(\alpha\varpi_1 + \varphi\varpi_2 + \gamma \sum_{i=1}^{n} \mu_i \varkappa_i\right) \leq \alpha f(\varpi_1) + \varphi f(\varpi_2) + \gamma \sum_{i=1}^{n} \mu_i f(\varkappa_i). \tag{4}$$

Remark 1. *From the inequality (4),*

1. *If we choose $\alpha = \varphi = 1$ and $\gamma = -1$, we obtain the inequality (3).*
2. *If we take $\alpha = \varphi = 0$ and $\gamma = 1$, then we have the inequality (2).*

In [14], Kian and Moslehian used the Jensen–Mercer inequality in 2013 to obtain the following Hermite–Hadamard–Mercer-type inequalities:

Let f be a convex function on $[\varpi_1, \varpi_2]$. Then

$$\begin{aligned} f\left(\varpi_1+\varpi_2-\frac{\varkappa_1+\varkappa_2}{2}\right) &\leq f(\varpi_1)+f(\varpi_2)-\int_0^1 f(\kappa\varkappa_1+(1-\kappa)\varkappa_2)d\kappa \\ &\leq f(\varpi_1)+f(\varpi_2)-f\left(\frac{\varkappa_1+\varkappa_2}{2}\right), \end{aligned}$$

and

$$\begin{aligned} f\left(\varpi_1+\varpi_2-\frac{\varkappa_1+\varkappa_2}{2}\right) &\leq \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2} f(\varpi_1+\varpi_2-\kappa)d\kappa \\ &\leq f(\varpi_1)+f(\varpi_2)-\frac{f(\varkappa_1)+f(\varkappa_2)}{2} \end{aligned}$$

for all $\varkappa_1, \varkappa_2 \in [\varpi_1, \varpi_2]$.

The majority of results and applications related to the Mercer inequality can be summarized as follows:

In [15], the researchers also derived novel variations of the Jensen–Mercer inequality connected to certain positive tuples and provided several related integral versions for various means. Moradi and Furuichi, in [16], proved some generalizations for convex functions in a different way. In [17], Ali et al. presented a new identity associated with twice-differentiable functions and used it to establish some new Hermite–Hadamard–Mercer inequalities. In [18], operator h-convex functions were introduced by Abbasi et al., who also proved operator versions of the Jensen and Jensen–Mercer-type inequalities for a few classes of operator h-convex functions.

Fractional calculus is a useful tool for interpreting both natural phenomena and common situations. The Riemann–Liouville integral is the most commonly used fractional integral operator. Riemann–Liouville fractional integrals are a generalized form of integral calculus and are used to define non-integer orders of derivatives of functions. The Hermite–Hadamard inequality often takes new forms and is the basis for several investigations in inequality theory. Sarıkaya and collaborators, in [4], were the first to develop an iteration of the Hermite–Hadamard inequality using the fractional integral operator. Since then, a combination of methods derived from both fractional integral calculus and convexity has been used to obtain classical inequalities with different fractional operators. Hermite–Hadamard-type inequalities involving Riemann–Liouville fractional integral operators for convex functions also appear frequently. The Riemann–Liouville fractional integral operator is defined as follows in [19]:

Definition 2. *Let $f \in L[\varpi_1, \varpi_2]$.The Riemann–Liouville integrals $J^{\alpha}_{\varpi_1^+} f$ and $J^{\alpha}_{\varpi_2^-} f$ of order $\alpha > 0$ with $\varpi_1 \geq 0$ are defined by*

$$J^{\alpha}_{\varpi_1^+} f(u) = \frac{1}{\Gamma(\alpha)}\int_{\varpi_1}^{u} (u-\kappa)^{\alpha-1} f(\kappa)d\kappa, \quad u > \varpi_1,$$

and

$$J^{\alpha}_{\varpi_2^-} f(u) = \frac{1}{\Gamma(\alpha)}\int_{u}^{\varpi_2} (\kappa-u)^{\alpha-1} f(\kappa)d\kappa, \quad u < \varpi_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function as its definition $\Gamma(\varsigma) = \int_0^{\infty} e^{-\kappa}\kappa^{\varsigma-1}d\kappa$. It is to be noted that $J^{\alpha}_{\varpi_1^+} f(\kappa) = J^{\alpha}_{\varpi_2^-} f(\kappa) = f(\kappa)$; the fractional integral simplifies to the classical integral when $\alpha = 1$.

Meanwhile, in [4], Sarikaya et al. gave the following interesting result for the Riemann–Liouville operators:

Theorem 1. *Let $f : [\varpi_1, \varpi_2] \to \mathbb{R}$ be a possitive function with $0 \leq \varpi_1 < \varpi_2$ and $\varpi_1, \varpi_2 \in I$, with $\varpi_1 < \varpi_2$. $f \in L[\varpi_1, \varpi_2]$. If f is a convex function on $[\varpi_1, \varpi_2]$, then the following inequality for fractional integrals holds*

$$f\left(\frac{\varpi_1+\varpi_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}\left[(J^{\alpha}_{\varpi_1^+}f(\varpi_2)+(J^{\alpha}_{\varpi_2^-}f(\varpi_1)\right] \leq \frac{f(\varpi_1)+f(\varpi_2)}{2}$$

The Jensen–Mercer inequality has been the subject of much research recently, particularly with the use of fractional integral operators. Some of these works are as follows:

In [20,21], the authors proved new Hermite–Hadamard–Mercer-type inequalities for the Riemann–Liouville and $k-$Riemann–Liouville integrals, respectively. The fractional integral version of the Hermite–Hadamard–Mercer inequality was initially discovered by researchers in reference [20]. Numerous novel results of the Mercer type were obtained as a result of this investigation. Also, Butt et al. obtained new Hermite–Hadamard–Mercer-type results using different types of convexities in [22]. In their study, Sitthiwirattham and colleagues demonstrated the existence of a novel identity (Lemma 8) for fractional integration, which is analogous to the Hermite–Hadamard–Mercer-type inequality presented in [23]. They presented some graphical examples of the inequalities obtained to show the validity of their results. In addition to the aforementioned studies, a multitude of other studies of inequalities of the Hermite–Hadamard–Mercer type have been conducted, with some of these being referenced in the bibliography [24–28].

In [29], Raina defined a class of clearly specified functions known as Raina functions using

$$F^{\sigma}_{\wp,\lambda}(\kappa) = F^{\sigma(0),\sigma(1)\cdots}_{\wp,\lambda}(\kappa) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\wp k+\lambda)}\kappa^k, \quad \wp,\lambda>0, |\kappa|<\mathbb{R} \tag{5}$$

where the coefficent $\sigma(k)$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, is a bounded sequence of positive real numbers and $\mathbb{R}$ is the real number. With the help of (5) , Raina and Agarwal et al., in [30], defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$J^{\sigma}_{\wp,\lambda,\varpi_1^+;w}f(\kappa) = \int_{\varpi_1}^{\kappa}(\kappa-u)^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\kappa-u)^{\wp}]f(u)du, \quad \kappa>\varpi_1 \tag{6}$$

and

$$J^{\sigma}_{\wp,\lambda,\varpi_2^-;w}f(\kappa) = \int_{\kappa}^{\varpi_2}(u-\kappa)^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(u-\kappa)^{\wp}]f(u)du, \quad \kappa<\varpi_2 \tag{7}$$

where $\wp,\lambda>0, w\in\mathbb{R}$ and $f(u)$ are such that the integrals on the right side exist.

The Raina function has been employed with considerable frequency in recent times. This is largely attributable to its relationship with fractional operators. Following the establishment of the (6) and (7) fractional operators, this special function was applied to a multitude of significant inequalities, resulting in the discovery of new results, definitions, generalizations, and even novel operators. Those interested in further investigating this function are invited to examine the studies referenced in [31–35].

In [36], the researchers presented a novel form of Hermite–Hadamard inequality by utilizing the Raina function for fractional operators.

Theorem 2. *Let $f : [\varpi_1, \varpi_2] \to \mathbb{R}$ be a convex function on $[\varpi_1, \varpi_2]$, with $\varpi_1 < \varpi_2$; then, the following inequalities for fractional integrals hold*

$$\begin{aligned} & f\left(\frac{\varpi_1+\varpi_2}{2}\right) \\ \leq\ & \frac{1}{2(\varpi_2-\varpi_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\varpi_2-\varpi_1)^{\wp}]}\left[(J^{\sigma}_{\wp,\lambda,\varpi_1^+;w}f(\varpi_2)+J^{\sigma}_{\wp,\lambda,\varpi_2^-;w}f(\varpi_1)\right] \\ \leq\ & \frac{f(\varpi_1)+f(\varpi_2)}{2} \end{aligned} \tag{8}$$

with $\lambda > 0$.

The aim of this study is to propose new generalizations of the Hermite–Hadamard–Mercer type based on fractional operators obtained for the Raina function. Furthermore, a new lemma of this type is demonstrated, and new identities are derived by employing the definition of a convex function. The new and existing results in the literature are shown for special values.

2. Main Results

This section presents our main results, which are a generalization of fractional operators and a critical analysis of their significance in the context of the existing literature.

Theorem 3. *$f : [\omega_1, \omega_2] \to \mathbb{R}$ is a convex function. Let $\alpha, \varphi \in [0,1]$,$\gamma \in (0,1]$ be coefficients of sums $\alpha + \varphi + \gamma = 1$ and $\lambda > 0$, meaning that the following inequality*

$$\begin{aligned} & f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{x_1+x_2}{2}\right) \\ \leq\ & \frac{1}{2(\gamma x_2 - \gamma x_1)^{\lambda} F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2 - \gamma x_1)^{\wp}]} \\ & \times\left[J^{\sigma}_{\wp,\lambda,(\alpha\omega_1+\varphi\omega_2+\gamma x_1)^{+};w} f(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2) + J^{\sigma}_{\wp,\lambda,(\alpha\omega_1+\varphi\omega_2+\gamma x_2)^{-};w} f(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1)\right] \\ \leq\ & \left[\alpha f(\omega_1) + \varphi f(\omega_2) + \gamma\frac{f(x_1)+f(x_2)}{2}\right] \end{aligned} \tag{9}$$

holds for all $x_1, x_2 \in [\omega_1, \omega_2]$ with $x_1 < x_2$.

Proof. We first note that if f is a convex function, then

$$\begin{aligned} f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{\omega_1+\omega_2}{2}\right) &= f\left(\frac{\alpha\omega_1 + \varphi\omega_2 + \gamma\omega_1 + \alpha\omega_1 + \varphi\omega_2 + \gamma\omega_2}{2}\right) \\ &\leq \frac{1}{2}[f(\alpha\omega_1 + \varphi\omega_2 + \gamma\omega_1) + f(\alpha\omega_1 + \varphi\omega_2 + \gamma\omega_2)]. \end{aligned}$$

Secondly, using change of variable, for $\alpha\omega_1 + \varphi\omega_2 + \gamma\omega_1 = \kappa(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1) + (1-\kappa)(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2)$ and $\alpha\omega_1 + \varphi\omega_2 + \gamma\omega_2 = (1-\kappa)(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1) + \kappa(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2)$ with $\kappa \in [0,1]$,we obtain

$$\begin{aligned} & f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{x_1+x_2}{2}\right) \\ \leq\ & \frac{1}{2}[f(\kappa(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1) + (1-\kappa)(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2)) \\ & + f(\kappa(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2) + (1-\kappa)(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1))]. \end{aligned} \tag{10}$$

Subsequently, by multiplying both sides of (10) by $\kappa^{\lambda-1} F^{\sigma}_{\wp,\lambda}[w(\gamma x_2 - \gamma x_1)^{\wp}\kappa^{\wp}]$ and integrating the resulting identity with respect to κ over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{x_1+x_2}{2}\right)\kappa^{\lambda-1} F^{\sigma}_{\wp,\lambda}[w(\gamma x_2 - \gamma x_1)^{\wp}\kappa^{\wp}]d\kappa \\ \leq\ & \frac{1}{2}\left[\int_0^1 \kappa^{\lambda-1} F^{\sigma}_{\wp,\lambda}[w(\gamma x_2 - \gamma x_1)^{\wp}\kappa^{\wp}] f(\kappa(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1) + (1-\kappa)(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2))d\kappa\right. \\ & \left. + \int_0^1 \kappa^{\lambda-1} F^{\sigma}_{\wp,\lambda}[w(\gamma x_2 - \gamma x_1)^{\wp}\kappa^{\wp}] f(\kappa(\alpha\omega_1 + \varphi\omega_2 + \gamma x_2) + (1-\kappa)(\alpha\omega_1 + \varphi\omega_2 + \gamma x_1))d\kappa\right]. \end{aligned}$$

By means of a straightforward analysis calculation (namely change of variable), the following inequality is derived:

$$
\begin{aligned}
& f\left(\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{x_1+x_2}{2}\right)F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}] \\
\leq\ & \frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda}} \\
& \times\left[\int_{\alpha\varpi_1+\varphi\varpi_2+\gamma x_1}^{\alpha\varpi_1+\varphi\varpi_2+\gamma x_2}[(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)-u]^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w[(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)-u]^{\wp}]f(u)du\right. \\
& \left.+\int_{\alpha\varpi_1+\varphi\varpi_2+\gamma x_1}^{\alpha\varpi_1+\varphi\varpi_2+\gamma x_2}[u-(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)]^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w[u-(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)]^{\wp}]f(u)du\right].
\end{aligned}
$$

Using identities (6) and (7), we have

$$
\begin{aligned}
& f\left(\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{x_1+x_2}{2}\right) \\
\leq\ & \frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]} \\
& \times\left[J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)\right].
\end{aligned}
$$

Thus, the first inequality is obtained in Theorem 3. To prove the second inequality in (9) from the convexity of f, we can write

$$
\begin{aligned}
& f(\kappa(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+(1-\kappa)(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)) \\
\leq\ & \kappa f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+(1-\kappa)f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
& f(\kappa(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+(1-\kappa)(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)) \\
\leq\ & \kappa f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+(1-\kappa)f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1).
\end{aligned}
\tag{12}
$$

Adding inequalities (11) and (12), and using the Jensen–Mercer inequality, we have

$$
\begin{aligned}
& f(\kappa(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+(1-\kappa)(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)) \\
& +f(\kappa(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+(1-\kappa)(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)) \\
\leq\ & f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2) \\
\leq\ & 2\alpha f(\varpi_1)+2\varphi f(\varpi_2)+\gamma[f(x_1)+f(x_2)]
\end{aligned}
$$

Then, multiplying both sides of obtained inequality by $\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]$, and integrating with respect to κ over $[0,1]$, we have

$$
\begin{aligned}
& \int_0^1\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]f(\kappa(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+(1-\kappa)(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2))d\kappa \\
& +\int_0^1\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]f(\kappa(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+(1-\kappa)(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1))d\kappa \\
\leq\ & \int_0^1\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}][2\alpha f(\varpi_1)+2\varphi f(\varpi_2)+\gamma[f(x_1)+f(x_2)]]d\kappa \\
=\ & \left(\alpha f(\varpi_1)+\varphi f(\varpi_2)+\gamma\frac{f(x_1)+f(x_2)}{2}\right)F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}].
\end{aligned}
$$

A simple calculation and using change of variable, the above inequality yields the second inequality of our first result, thereby concluding the proof. □

Remark 2. *In Theorem 3, the following points are true:*

1. *If we choose $\alpha = 0 = \varphi$ and $\gamma = 1$, then we have the inequality (8), which is the Hermite–Hadamard inequality for the Raina function.*
2. *If we choose $\alpha = 0 = \varphi$, $\gamma = 1$ and $\sigma(0) = 1, w = 0$, then we have the Hermite–Hadamard inequality for the Reimann–Liouville fractional integral operator as follows:*

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{\Gamma(\lambda+1)}{2^{\lambda}(x_2-x_1)^{\lambda}}\left[J^{\lambda}_{x_1^+}f(x_2) + J^{\lambda}_{x_2^-}f(x_1)\right] \leq \frac{f(x_1)+f(x_2)}{2}.$$

Also, if we take $\lambda = 1$ in the above inequality, obviously, we have the Hermite–Hadamard inequality.

Theorem 4. *$f : [\omega_1, \omega_2] \to \mathbb{R}$ is a convex function. Let $\alpha, \varphi \in [0,1], \gamma \in (0,1]$ be coefficients of sums $\alpha + \varphi + \gamma = 1$ and $\lambda > 0$, and then we obtain the following inequality:*

$$\begin{aligned} & f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{x_1+x_2}{2}\right) \\ \leq\ & \frac{(\hbar+1)^{\lambda}}{2(\gamma x_2 - \gamma x_1)^{\lambda} F^{\sigma}_{\wp,\lambda+1}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\right]} \\ & \times\left[J^{\sigma}_{\wp,\lambda,\left(\alpha\omega_1+\varphi\omega_2+\gamma\frac{x_1+\hbar x_2}{2}\right)^+;w} f(\alpha\omega_1+\varphi\omega_2+\gamma x_2)\right. \\ & \left. + J^{\sigma}_{\wp,\lambda,\left(\alpha\omega_1+\varphi\omega_2+\gamma\frac{\hbar x_1+x_2}{2}\right)^-;w} f(\alpha\omega_1+\varphi\omega_2+\gamma x_1)\right] \\ \leq\ & \left[\alpha f(\omega_1) + \varphi f(\omega_2) + \gamma\frac{f(x_1)+f(x_2)}{2}\right] \end{aligned}$$

where $x_1, x_2 \in [\omega_1, \omega_2]$ with $x_1 < x_2$ and $\hbar \in \mathbb{N}$.

Proof. Since f is a convex function, we can write

$$f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{\omega_1+\omega_2}{2}\right) \leq \frac{1}{2}[f(\alpha\omega_1+\varphi\omega_2+\gamma\omega_1) + (\alpha\omega_1+\varphi\omega_2+\gamma\omega_2)].$$

Similarly to the first theorem, using change of variable, for $\omega_1 = \frac{\kappa}{\hbar+1}x_1 + \frac{\hbar+1-\kappa}{\hbar+1}x_2$ and $\omega_2 = \frac{\hbar+1-\kappa}{\hbar+1}x_1 + \frac{\kappa}{\hbar+1}x_2$ with $\kappa \in [0,1]$, we obtain

$$\begin{aligned} & f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\frac{x_1+x_2}{2}\right) \\ \leq\ & \frac{1}{2}\left[f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\left(\frac{\kappa}{\hbar+1}x_1 + \frac{\hbar+1-\kappa}{\hbar+1}x_2\right)\right)\right. \\ & \left. + f\left(\alpha\omega_1 + \varphi\omega_2 + \gamma\left(\frac{\hbar+1-\kappa}{\hbar+1}x_1 + \frac{\kappa}{\hbar+1}x_2\right)\right)\right]. \end{aligned} \tag{13}$$

Multiplying both sides of (13) by $\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\kappa^{\wp}\right]$ and integrating the resulting inequality with respect to κ over $[0,1]$, we have

$$\begin{aligned} & \int_0^1 \kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\kappa^{\wp}\right] f\left(\alpha\omega_1+\varphi\omega_2+\gamma\frac{x_1+x_2}{2}\right)d\kappa \\ \leq\ & \frac{1}{2}\left[\int_0^1 \kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\kappa^{\wp}\right] f\left(\alpha\omega_1+\varphi\omega_2+\gamma\left(\frac{\kappa}{\hbar+1}x_1+\frac{\hbar+1-\kappa}{\hbar+1}x_2\right)\right)d\kappa\right. \\ & \left. + \int_0^1 \kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\kappa^{\wp}\right] f\left(\alpha\omega_1+\varphi\omega_2+\gamma\left(\frac{\hbar+1-\kappa}{\hbar+1}x_1+\frac{\kappa}{\hbar+1}x_2\right)\right)d\kappa\right]. \end{aligned}$$

So, from identities (6) and (7) and the change of variable, we can write

$$
\begin{aligned}
& f\left(\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{x_1+x_2}{2}\right)F^{\sigma}_{\wp,\lambda+1}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\right] \\
\leq\ & \frac{(\hbar+1)^{\lambda}}{2(\gamma x_2-\gamma x_1)^{\lambda}} \\
& \times\left[\int_{\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{x_1+\hbar x_2}{2}}^{\alpha\varpi_1+\varphi\varpi_2+\gamma x_2}\left[(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)-u\right]^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w[(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)-u]^{\wp}]f(u)du\right. \\
& \left.+\int_{\alpha\varpi_1+\varphi\varpi_2+\gamma x_1}^{\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{\hbar x_1+x_2}{2}}\left[u-(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)\right]^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w[u-(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)]^{\wp}]f(u)du\right] \\
=\ & \frac{(\hbar+1)^{\lambda}}{2(\gamma x_2-\gamma x_1)^{\lambda}}\left[J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{x_1+\hbar x_2}{2})^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right. \\
& \left.+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma\frac{\hbar x_1+x_2}{2})^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)\right].
\end{aligned}
$$

Thus, the first inequality is proved in Theorem 4.

Now, to prove the second inequality, utilizing the Jensen–Mercer inequality, we can write

$$
\begin{aligned}
& f\left(\alpha\varpi_1+\varphi\varpi_2+\gamma\left(\frac{\kappa}{\hbar+1}x_1+\frac{\hbar+1-\kappa}{\hbar+1}x_2\right)\right) \\
\leq\ & \alpha f(\varpi_1)+\varphi f(\varpi_2)+\gamma\left(\frac{\kappa}{\hbar+1}f(x_1)+\frac{\hbar+1-\kappa}{\hbar+1}f(x_2)\right)
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
& f\left(\alpha\varpi_1+\varphi\varpi_2+\gamma\left(\frac{\hbar+1-\kappa}{\hbar+1}x_1+\frac{\kappa}{\hbar+1}x_2\right)\right) \\
\leq\ & \alpha f(\varpi_1)+\varphi f(\varpi_2)+\gamma\left(\frac{\hbar+1-\kappa}{\hbar+1}f(x_1)+\frac{\kappa}{\hbar+1}f(x_2)\right)
\end{aligned}
\tag{15}
$$

Similarly to the first theorem, adding inequalities (14) and (15), multiplying both sides of the obtained inequality by $\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}\left[w\left(\frac{\gamma x_2-\gamma x_1}{\hbar+1}\right)^{\wp}\kappa^{\wp}\right]$, and integrating with respect to κ over $[0,1]$, we have the second inequality. This completes the proof. □

Corollary 1. *If we choose $\alpha=0=\varphi$ and $\gamma=1$ in Theorem 4, then we obtain the Hermite–Hadamard-type inequality for the Raina function;*

$$
\begin{aligned}
& f\left(\frac{x_1+x_2}{2}\right) \\
\leq\ & \frac{(\hbar+1)^{\lambda}}{2(x_2-x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}\left[w\left(\frac{x_2-x_1}{\hbar+1}\right)^{\wp}\right]} \\
& \times\left[J^{\sigma}_{\wp,\lambda,(\frac{x_1+\hbar x_2}{2})^{+};w}f(x_2)+J^{\sigma}_{\wp,\lambda,(\frac{\hbar x_1+x_2}{2})^{-};w}f(x_1)\right] \\
\leq\ & \left[\frac{f(x_1)+f(x_2)}{2}\right].
\end{aligned}
$$

Also, if we take $\hbar = 1$, we have

$$\begin{aligned} & f\left(\frac{x_1+x_2}{2}\right) \\ \leq\ & \frac{2^{\lambda-1}}{(x_2-x_1)^{\lambda} F^{\sigma}_{\wp,\lambda+1}\left[w\left(\frac{x_2-x_1}{2}\right)^{\wp}\right]} \\ & \times\left[J^{\sigma}_{\wp,\lambda,\left(\frac{x_1+x_2}{2}\right)^{+};w} f(x_2) + J^{\sigma}_{\wp,\lambda,\left(\frac{x_1+x_2}{2}\right)^{-};w} f(x_1)\right] \\ \leq\ & \left[\frac{f(x_1)+f(x_2)}{2}\right]. \end{aligned}$$

Corollary 2. *If we choose $\alpha = 0 = \varphi, \gamma = 1$ and $\sigma(0) = 1, w = 0$ in Theorem 4, then we obtain*

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{\Gamma(\lambda+1)(\hbar+1)^{\lambda}}{2(x_2-x_1)^{\lambda}}\left[J^{\lambda}_{\left(\frac{x_1+\hbar x_2}{2}\right)^{+}} f(x_2) + J^{\lambda}_{\left(\frac{\hbar x_1+x_2}{2}\right)^{-}} f(x_1)\right] \leq \frac{f(x_1)+f(x_2)}{2}.$$

Also, if we take $\hbar = 1$, we have

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{2^{\lambda-1}\Gamma(\lambda+1)}{(x_2-x_1)^{\lambda}}\left[J^{\lambda}_{\left(\frac{x_1+x_2}{2}\right)^{+}} f(x_2) + J^{\lambda}_{\left(\frac{x_1+x_2}{2}\right)^{-}} f(x_1)\right] \leq \frac{f(x_1)+f(x_2)}{2}$$

in [37].

Corollary 3. *In Corollary 2, if we choose $\lambda = 1$, we have*

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{(\hbar+1)}{(x_2-x_1)}\int_{x_1}^{x_2} f(\kappa)d\kappa \leq \frac{f(x_1)+f(x_2)}{2}.$$

3. Further Generalized Results

Now, we obtain some new general results related to Hermite–Hadamard–Mercer-type inequalities using Lemma 1 and some elementary methods.

Lemma 1. *Let $f : [\omega_1, \omega_2] \to \mathbb{R}$ be a differentiable function on (ω_1, ω_2). If $f' \in L[\omega_1, \omega_2]$ and $\alpha, \varphi \in [0,1], \gamma \in (0.1]$ are coefficients of sums $\alpha + \varphi + \gamma = 1$ and $\lambda > 0$, then we have the following identity;*

$$\begin{aligned} & \frac{(f(\alpha\omega_1+\varphi\omega_2+\gamma x_1)+f(\alpha\omega_1+\varphi\omega_2+\gamma x_2))}{2} - \frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda} F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]} \\ & \times\left[J^{\sigma}_{\wp,\lambda,(\alpha\omega_1+\varphi\omega_2+\gamma x_2)^{-};w} f(\alpha\omega_1+\varphi\omega_2+\gamma x_1) + J^{\sigma}_{\wp,\lambda,(\alpha\omega_1+\varphi\omega_2+\gamma x_1)^{+};w} f(\alpha\omega_1+\varphi\omega_2+\gamma x_2)\right] \\ = \ & \frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]} \\ & \times\left[\int_0^1 (1-\kappa)^{\lambda} F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}(1-\kappa)^{\wp}] f'(\alpha\omega_1+\varphi\omega_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa\right. \\ & \left. - \int_0^1 \kappa^{\lambda} F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}] f'(\alpha\omega_1+\varphi\omega_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa\right] \end{aligned} \tag{16}$$

for all $x_1, x_2 \in [\omega_1, \omega_2]$ with $x_1 < x_2$.

Proof. In this step, we apply integration by parts to the integrals on the right-hand side of Equation (16), and then we have

$$\begin{aligned}
&\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]} \\
&\times\left[\int_0^1(1-\kappa)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}(1-\kappa)^{\wp}]f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa\right. \\
&\left.-\int_0^1\kappa^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa\right] \\
=\;&\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}[\mathcal{M}_1-\mathcal{M}_2].
\end{aligned} \tag{17}$$

As a consequence, from the change of variable, we then have following results:

$$\begin{aligned}
\mathcal{M}_1 =\;&\int_0^1(1-\kappa)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}(1-\kappa)^{\wp}]f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa \\
=\;&-\frac{(1-\kappa)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}(1-\kappa)^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)}{(\gamma x_2-\gamma x_1)}\Bigg|_0^1 \\
&-\frac{1}{(\gamma x_2-\gamma x_1)}\int_0^1(1-\kappa)^{\lambda-1}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}(1-\kappa)^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa \\
=\;&\frac{F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)}{(\gamma x_2-\gamma x_1)} \\
&-\frac{1}{(\gamma x_2-\gamma x_1)}\int_{\alpha\varpi_1+\varphi\varpi_2+\gamma x_1}^{\alpha\varpi_1+\varphi\varpi_2+\gamma x_2}\left(\frac{u-(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)}{\gamma x_2-\gamma x_1}\right)^{\lambda-1} \\
&\qquad\times F^{\sigma}_{\wp,\lambda+1}[w(u-(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1))^{\wp}]f(u)\frac{du}{(\gamma x_2-\gamma x_1)} \\
=\;&\frac{F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)}{(\gamma x_2-\gamma x_1)} \\
&-\frac{1}{(\gamma x_2-\gamma x_1)}\frac{1}{(\gamma x_2-\gamma x_1)^{\lambda}}J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^-;w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathcal{M}_2 =\;&\int_0^1\kappa^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa \\
=\;&\frac{\kappa^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)}{(\gamma x_2-\gamma x_1)}\Bigg|_0^1 \\
&-\frac{1}{(\gamma x_2-\gamma x_1)}\int_0^1\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)d\kappa \\
=\;&\frac{F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_1)}{(\gamma x_2-\gamma x_1)} \\
&+\frac{1}{(\gamma x_2-\gamma x_1)}\frac{1}{(\gamma x_2-\gamma x_1)^{\lambda}}J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^+;w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2).
\end{aligned}$$

Substituting the values of $\mathcal{M}_1$ and $\mathcal{M}_2$ in (17), we obtain the required results. □

Corollary 4. *If we choose $\alpha=0=\varphi$ and $\gamma=1$ in Lemma 1, then we have a different identity, defined as follows:*

$$\begin{aligned}
&\frac{f(x_1)+f(x_2)}{2}-\frac{1}{2(x_2-x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]}\left[J^{\sigma}_{\wp,\lambda,x_2^-;w}f(x_1)+J^{\sigma}_{\wp,\lambda,x_1^+;w}f(x_2)\right]\\
=\;&\frac{(x_2-x_1)}{2}\left[\int_0^1(1-\kappa)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}(1-\kappa)^{\wp}]f'((\kappa x_1+(1-\kappa)x_2)d\kappa\right.\\
&\left.-\int_0^1\kappa^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}\kappa^{\wp}]f'((\kappa x_1+(1-\kappa)x_2)d\kappa\right].
\end{aligned}$$

Theorem 5. *Under the assumptions of Lemma 1, if $|f'|$ is a convex function, we reach the following conclusion:*

$$\begin{aligned}
&\left|\frac{(f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_2))}{2}-\frac{1}{(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\right.\\
&\left.\times\left[J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^-;w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^+;w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right]\right|\\
\leq\;&\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\\
&\times\left[\left(2\alpha\left|f'(\varpi_1)\right|+2\varphi\left|f'(\varpi_2)\right|\right)\left(F^{\sigma_1}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]-F^{\sigma_2}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]\right)\right.\\
&+\left(\left|f'(x_1)\right|+\left|f'(x_2)\right|\right)\left(F^{\sigma_3}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]-F^{\sigma_4}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]\right)\\
&\left.+\left(\left|f'(x_1)\right|+\left|f'(x_2)\right|\right)\left(F^{\sigma_5}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]-F^{\sigma_6}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &= \sigma(k)\left(1-\left(\frac{1}{2}\right)^{c+1}\right)\\
\sigma_2 &= \sigma(k)\left(\frac{1}{2}\right)^{c+1}\\
\sigma_3 &= \sigma(k)\frac{2^{-c-2}(2^{c+2}-c-3)}{c^2+3c+2}\\
\sigma_4 &= \sigma(k)\frac{\left(\frac{1}{2}\right)^{c+2}}{c+2}\\
\sigma_5 &= \sigma(k)\left(\frac{1-\left(\frac{1}{2}\right)^{c+2}}{c+2}\right)\\
\sigma_6 &= \sigma(k)\frac{2^{-c-2}(c+3)}{c^2+3c+2}\\
c &= \lambda+\wp k-1.
\end{aligned}$$

Proof. By utilizing Lemma 1, the property of absolute value and convexity of $|f'|$, we obtain

$$
\begin{aligned}
&\left|\frac{(f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_2))}{2}-\frac{1}{(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\right.\\
&\left.\times\left[J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right]\right|\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\int_0^1\left|(1-\kappa)^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\gamma x_2-\gamma x_1)^{\wp}(1-\kappa)^{\wp}]-\kappa^{\lambda-1}F^{\sigma}_{\wp,\lambda}[w(\gamma x_2-\gamma x_1)^{\wp}\kappa^{\wp}]\right|\\
&\times\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|d\kappa]\\
=\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\left.\times\int_0^1\left|(1-\kappa)^{\lambda+\wp k}-\kappa^{\lambda+\wp k}\right|\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|d\kappa\right]\\
=\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\times\left\{\int_0^{\frac{1}{2}}\left[(1-\kappa)^{\lambda+\wp k}-\kappa^{\lambda+\wp k}\right]\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|d\kappa\right.\\
&\left.\left.+\int_{\frac{1}{2}}^1\left[\kappa^{\lambda+\wp k}-(1-\kappa)^{\lambda+\wp k}\right]\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|d\kappa\right\}\right]\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\times\left\{\int_0^{\frac{1}{2}}\left[(1-\kappa)^{\lambda+\wp k}-\kappa^{\lambda+\wp k}\right]\left(\alpha\left|f'(\varpi_1)\right|+\varphi\left|f'(\varpi_2)\right|+\gamma(\kappa\left|f'(x_1)\right|+(1-\kappa)\left|f'(x_2)\right|\right)d\kappa\right.\\
&\left.\left.+\int_{\frac{1}{2}}^1\left[\kappa^{\lambda+\wp k}-(1-\kappa)^{\lambda+\wp k}\right]\left(\alpha\left|f'(\varpi_1)\right|+\varphi\left|f'(\varpi_2)\right|+\gamma(\kappa\left|f'(x_1)\right|+(1-\kappa)\left|f'(x_2)\right|\right)d\kappa\right\}\right].
\end{aligned}
$$

After simple calculations and using the Raina function, we obtain the required result. □

Corollary 5. *If we choose $\alpha=0=\varphi$ and $\gamma=1$ in Theorem 5, then we obtain*

$$
\begin{aligned}
&\left|\frac{f(x_1)+f(x_2)}{2}-\frac{1}{(x_2-x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]}\left(J^{\sigma}_{\wp,\lambda,x_2{}^{-};w}f(x_1)+J^{\sigma}_{\wp,\lambda,x_1{}^{+};w}f(x_2)\right)\right|\\
\leq\ &\frac{(x_2-x_1)}{F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]}\frac{(|f'(x_1)|+|f'(x_2)|)}{2}\\
&\times\left[\left(F^{\sigma_3}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]-F^{\sigma_4}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]\right)\right.\\
&\left.+\left(F^{\sigma_5}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]-F^{\sigma_6}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]\right)\right].
\end{aligned}
$$

Theorem 6. *Under the assumptions of Lemma 1, if $|f'|^q$ is a convex function, then we obtain*

$$
\begin{aligned}
&\left|\frac{(f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_1))}{2}-\frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\right.\\
&\left.\times\left(J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right)\right|\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2}\frac{F^{\sigma_7}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}{F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\\
&\times\left\{\left(\frac{(\alpha|f'(\varpi_1)|^q+\varphi|f'(\varpi_2)|^q)}{2}+\gamma\left(\frac{|f'(x_1)|^q+3|f'(x_2)|^q}{8}\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{(\alpha|f'(\varpi_1)|^q+\varphi|f'(\varpi_2)|^q)}{2}+\gamma\left(\frac{3|f'(x_1)|^q+|f'(x_2)|^q}{8}\right)\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$\sigma_7=\sigma(k)\left(\frac{1-2^{-p(\lambda+\wp k)}}{p(\lambda+\wp k)+1}\right)^{\frac{1}{p}}$$

with $\frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$.

Proof. By utilizing Lemma 1 and Hölder's inequality, we obtain

$$
\begin{aligned}
&\left|\frac{(f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_1))}{2}-\frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\right.\\
&\left.\times\left(J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right)\right|\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\times\left\{\int_0^{\frac{1}{2}}\left[(1-\kappa)^{\lambda+\wp k}-\kappa^{\lambda+\wp k}\right]\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|d\kappa\right.\\
&\left.\left.+\int_{\frac{1}{2}}^1\left[\kappa^{\lambda+\wp k}-(1-\kappa)^{\lambda+\wp k}\right]\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|d\kappa\right\}\right]\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left[(1-\kappa)^{\lambda+\wp k}-\kappa^{\lambda+\wp k}\right]^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|^q d\kappa\right)^{\frac{1}{q}}\right.\right.\\
&\left.\left.\left.+\left(\int_{\frac{1}{2}}^1\left[\kappa^{\lambda+\wp k}-(1-\kappa)^{\lambda+\wp k}\right]^p d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|^q d\kappa\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}
$$

Here, if we use $(\mathcal{X}-\mathcal{Y})^{\delta}\leq\mathcal{X}^{\delta}-\mathcal{Y}^{\delta}$ for any $\mathcal{X}>\mathcal{Y}\geq 0$ and $\delta\geq 1$, then

$$
\begin{aligned}
&\left|\frac{(f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_1))}{2}-\frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\right.\\
&\left.\times\left(J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right)\right|\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left[(1-\kappa)^{p(\lambda+\wp k)}-\kappa^{p(\lambda+\wp k)}\right]d\kappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|^q d\kappa\right)^{\frac{1}{q}}\right.\right.\\
&\left.\left.\left.+\left(\int_{\frac{1}{2}}^1\left[\kappa^{p(\lambda+\wp k)}-(1-\kappa)^{p(\lambda+\wp k)}\right]d\kappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(\alpha\varpi_1+\varphi\varpi_2+\gamma(\kappa x_1+(1-\kappa)x_2)\right|^q d\kappa\right)^{\frac{1}{q}}\right\}\right].
\end{aligned}
$$

By utilizing the Jensen–Mercer inequality because of convexity of $|f'|^q$, we have

$$
\begin{aligned}
&\left|\frac{(f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)+f(\alpha\varpi_1+\varphi\varpi_2+\gamma\kappa x_1))}{2}-\frac{1}{2(\gamma x_2-\gamma x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\right.\\
&\left.\times\left(J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)^{-};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)+J^{\sigma}_{\wp,\lambda,(\alpha\varpi_1+\varphi\varpi_2+\gamma x_1)^{+};w}f(\alpha\varpi_1+\varphi\varpi_2+\gamma x_2)\right)\right|\\
\leq\ &\frac{(\gamma x_2-\gamma x_1)}{2F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\left[\sum_{k=0}^{\infty}\frac{\sigma_7(k)[w(\gamma x_2-\gamma x_1)^{\wp}]^k}{\Gamma(\wp k+\lambda+1)}\right.\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left[(1-\kappa)^{p(\lambda+\wp k)}-\kappa^{p(\lambda+\wp k)}\right]d\kappa\right)^{\frac{1}{p}}\right.\\
&\qquad\times\left(\int_0^{\frac{1}{2}}\left(\alpha\left|f'(\varpi_1)\right|^q+\varphi\left|f'(\varpi_2)\right|^q+\gamma(\kappa\left|f'(x_1)\right|^q+(1-\kappa)\left|f'(x_2)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^1\left[\kappa^{p(\lambda+\wp k)}-(1-\kappa)^{p(\lambda+\wp k)}\right]d\kappa\right)^{\frac{1}{p}}\\
&\qquad\left.\left.\times\left(\int_{\frac{1}{2}}^1\left(\alpha\left|f'(\varpi_1)\right|^q+\varphi\left|f'(\varpi_2)\right|^q+\gamma(\kappa\left|f'(x_1)\right|^q+(1-\kappa)\left|f'(x_2)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\right\}\right]\\
=\ &\frac{(\gamma x_2-\gamma x_1)}{2}\frac{F^{\sigma_7}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}{F^{\sigma}_{\wp,\lambda+1}[w(\gamma x_2-\gamma x_1)^{\wp}]}\\
&\times\left\{\left(\frac{(\alpha|f'(\varpi_1)|^q+\varphi|f'(\varpi_2)|^q)}{2}+\gamma\left(\frac{|f'(x_1)|^q+3|f'(x_2)|^q}{8}\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{(\alpha|f'(\varpi_1)|^q+\varphi|f'(\varpi_2)|^q)}{2}+\gamma\left(\frac{3|f'(x_1)|^q+|f'(x_2)|^q}{8}\right)\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where

$$
\left(\int_0^{\frac{1}{2}}\left[(1-\kappa)^{p(\lambda+\wp k)}-\kappa^{p(\lambda+\wp k)}\right]d\kappa\right)^{\frac{1}{p}}=\left(\int_{\frac{1}{2}}^1\left[\kappa^{p(\lambda+\wp k)}-(1-\kappa)^{p(\lambda+\wp k)}\right]d\kappa\right)^{\frac{1}{p}}=\left(\frac{1-2^{-p(\lambda+\wp k)}}{p(\lambda+\wp k)+1}\right)^{\frac{1}{p}}.
$$

□

Corollary 6. *If we choose $\alpha=0=\varphi$ and $\gamma=1$ in Theorem 6, then we obtain*

$$\begin{aligned}
&\left|\frac{f(x_1)+f(x_2)}{2}-\frac{1}{2(x_2-x_1)^{\lambda}F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]}\left(J^{\sigma}_{\wp,\lambda,x_2^-;w}f(x_1)+J^{\sigma}_{\wp,\lambda,x_1^+;w}f(x_2)\right)\right| \\
\leq\ & \frac{(x_2-x_1)}{2}\frac{F^{\sigma_7}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]}{F^{\sigma}_{\wp,\lambda+1}[w(x_2-x_1)^{\wp}]} \\
&\times\left[\left(\frac{|f'(x_1)|^q+3|f'(x_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f'(x_1)|^q+|f'(x_2)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

4. Conclusions

The Raina function has been the subject of extensive study and development by researchers since its relationship with the fractional integral operator was established. A multitude of useful fractional integral operators can be derived by making specific choices regarding the coefficient $\sigma(k)$. This paper presents new generalizations and results for the Hermite–Hadamard–Mercer-type functions (namely Theorems 3 and 4). The results were obtained through the application of a variety of methodologies. Furthermore, new and existing results in the literature were obtained by giving special values of α, φ, and γ to the results obtained. In particular, researchers may discover novel and additional results by utilizing Corollary 4, which is a significant result of this study. The main motivation for this work is the existence of similar results in the literature. In this regard, the presented consequences and methods in this work many encourage further investigation in this field by researchers.

Author Contributions: Conceptualization, Ç.Y. and M.Ç.; Methodology, Ç.Y. and M.Ç.; Validation, Ç.Y.; Formal analysis, Z.Ç., Ç.Y. and M.Ç.; Investigation, Ç.Y. and M.Ç.; Resources, Ç.Y. and M.Ç.; Writing-original draft, Ç.Y. and M.Ç.; Writing-review & editing, Z.Ç., Ç.Y., M.Ç. and L.I.C.; Visualization, Ç.Y. and L.I.C.; Supervision, Ç.Y.; Project administration, Ç.Y. and M.Ç.; Funding acquisition, L.I.C. and D.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Breaz, D.; Yildiz, Ç.; Cotîrlă, L.I.; Rahman, G.; Yergöz, B. New Hadamard type inequalities for modified *h*-convex functions. *Fractal Fract.* **2023**, *7*, 216. [CrossRef]
2. Mohammed, P.O.; Sarikaya, M.Z. On generalized fractional integral inequalities for twice differentiable convex functions. *J. Comput. Appl. Math.* **2020**, *372*, 112740. [CrossRef]
3. Rashid, S.; Butt, S.I.; Kanwal, S.; Ahmad, H.; Wang, M.K. Quantum integral inequalities with respect to Raina's function via coordinated generalized $\psi-$convex functions with applications. *J. Funct. Spaces* **2021**, *4*, 6631474. [CrossRef]
4. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
5. Butt, S.I.; Nadeem, M.; Qaisar, S.; Akdemir, A.O.; Abdeljawad, T. Hermite-Jensen-Mercer type inequalities for conformable integrals and related results. *Adv. Differ. Equ.* **2020**, *2020*, 501. [CrossRef]
6. Ali, M.A.; Asjad, M.I.; Budak, H.; Faridi, W.A. On Ostrowski-Mercer inequalities for differentiable harmonically convex functions with applications. *Math. Methods Appliedsci.* **2023**, *46*, 8546–8559. [CrossRef]
7. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs; Victoria University: Wellington, New Zealand, 2000.
8. Mitrinović, D.S. *Analytic Inequalities*; Springer-Verlag: Berlin/Heidelberg, Germany; New York, NY, USA, 1970.
9. Dragomir, S.S.; Agarwal, R. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
10. Kavurmaci, H.; Avci, M.; Özdemir, M.E. New inequalities of Hermite-Hadamard type for convex functions with applications. *J. Inequalities Appl.* **2011**, *2011*, 86. [CrossRef]

11. Yildiz, Ç.; Özdemir, M.E. On generalized inequalities of Hermite-Hadamard type for convex functions. *Int. Anal. Appl.* **2017**, *14*, 52–63.
12. Mercer, A.M. A variant of Jensen's inequality. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 73.
13. Pavić, Z.: The Jensen–Mercer inequality with infinite convex combinations. *Math. Sci. Appl. E Notes*, **2019**, *7*, 19–27. [CrossRef]
14. Kian, M.; Moslehian, M.S. Refinements of the operator Jensen-Mercer inequality. *Electron. J. Linear Algebra*. **2013**, *26*, 742–753. [CrossRef]
15. Khan, M.A.; Pecaric, J. New refinements of the Jensen-Mercer inequality associated to positive n-tuples. *Armen. J. Math.* **2020**, *12*, 1–12.
16. Moradi, H.R.; Furuichi, S. Improvement and generalization of some Jensen-Mercer-type inequalities, *J. Math. Inequal.* **2020**, *14*, 377–383. [CrossRef]
17. Ali, M.A.; Sitthiwirattham, T.; Köbis, E.; Hanif, A. Hermite-Hadamard-Mercer Inequalities Associated with Twice-Differentiable Functions with Applications. *Axioms* **2024**, *13*, 114. [CrossRef]
18. Abbasi, M.; Morassaei, A.; Mirzapour, F. Jensen-Mercer Type Inequalities for Operator h-Convex Functions. *Bull. Iranian Math. Soc.* **2020**, *48*, 2441–2462. [CrossRef]
19. Miller, K.S.; Ross, B. *An Introduction to the Fractional calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
20. Öğulmüş, H.; Sarıkaya, M.Z. Hermite-Hadamard-Mercer type inequalities for fractional integrals. *Filomat* **2021**, *35*, 2425–2436. [CrossRef]
21. Ali, M.A.; Hussain, T.; Iqbal, M.Z.; Ejaz, F. Inequalities of Hermite-Hadamard-Mercer type for convex functions via k-fracttional integrals. *Inter. Jour. Math. Mod. Comp.* **2020**, *10*, 227–238
22. Butt, S.I.; Nasir, J.; Qaisar, S.; Abualnaja, K.M. $k-$ Fractional variants of Hermite-Mercer-type inequalities via $sNewYork$: $-$convexity with applications. *J. Funct. Spaces* **2021**, *8*, 5566360. [CrossRef]
23. Sitthiwirattham, T.; Vivas-Cortez, M.; Ali, M.A.; Budak, H.; Avcı, I. A study of fractional Hermite-Hadamard-Mercer inequalities for differentiable functions. *Fractals* **2024**, *32*, 2440016. [CrossRef]
24. Vivas-Cortez, M.; Awan, M.U.; Javed, M.Z.; Kashuri, A.; Noor, M.A.; Noor, K.I.; Vlora, A. Some new generalized k-fractional Hermite-Hadamard-Mercer type integral inequalities and their applications. *AIMS Math.* **2022**, *7*, 3203–3220. [CrossRef]
25. Vivas-Cortez, M.; Javed, M.Z.; Awan, M.U.; Noor, M.A.; Dragomir, S.S. Bullen-Mercer type inequalities with applications in numerical analysis. *Alex. Eng. J.* **2024**, *96*, 15–33. [CrossRef]
26. Faisal, S.; Khan, M.A.; Iqbal, S. Generalized Hermite-Hadamard-Mercer type inequalities via majorization. *Filomat* **2022**, *36*, 469–483. [CrossRef]
27. Sahoo, S.K.; Hamed, Y.S.; Mohammed, P.O.; Kodamasingh, B.; Nonlaopon, K. New midpoint type Hermite-Hadamard-Mercer inequalities pertaining to Caputo-Fabrizio fractional operators. *Alex. Eng. J.* **2023**, *65*, 689–698. [CrossRef]
28. Liu, J.B.; Butt, S.I.; Nasir, J.; Aslam, A.; Fahad, A.; Soontharanon, J. Jensen-Mercer variant of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional operator. *AIMS Math*, **2022**, *7*, 2123–2141. [CrossRef]
29. Raina, R.K. On generalized Wright's hypergeometric functions and fractional calculus operators. *East Asian Math. J.* **2005**, *21*, 191–203.
30. Agarwal, R.P.; Luo, M.J.; Raina, R.K. On Ostrowski Type Inequalities. *Fasc. Math.* **2016**, *56*, 5–27. [CrossRef]
31. Set, E.; Çelik, B.; Özdemir, M.E.; Aslan, M. Some New Results on Hermite-Hadamard-Mercer-Type Inequalities Using a General Family of Fractional Integral Operators. *Fractal Fract.* **2021**, *5*, 68. [CrossRef]
32. Set, E.; Noor, M.A.; Awan, M.U.; Gözpinar, A. Generalized Hermite-Hadamard type inequalities involving fractional integral operators. *J. Inequalities Appl.* **2017**, *169*, 1–10. [CrossRef]
33. Srivastava, H.M. A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics. *Symmetry* **2021**, *13*, 2294. [CrossRef]
34. Set, E.; Choi, J.; Gözpinar, A. Hermite-Hadamard type inequalities for the generalized $k-$fractional integral operators. *J. Inequalities Appl.* **2017**, *2017*, 206. [CrossRef] [PubMed]
35. Butt, S.I.; Nadeem, M.; Tariq, M.; Aslam, A. New integral type inequalities via Raina-convex functions and its applications. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2021**, *70*, 1011–1035. [CrossRef]
36. Yaldiz, H.; Sarikaya, M.Z. On the Hermite-Hadamard type inequalities for fractional integral operator. *Kragujev. J. Math.* **2020**, *44*, 369–378. [CrossRef]
37. Sarıkaya, M.Z.; Hüseyin, Y. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2017**, *17*, 1049–1059. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.