DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **12**, Number **3**, June **2019**

doi:10.3934/dcdss.2019033

pp. **503–512**

# ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM FOR SCHRÖDINGER EQUATION

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Abstract. In this paper, an optimal control problem for Schrödinger equation with complex coefficient which contains gradient is examined. A theorem is given that states the existence and uniqueness of the solution of the initial-boundary value problem for Schrödinger equation. Then for the solution of the optimal control problem, two different cases are investigated. Firstly, it is shown that the optimal control problem has a unique solution for $\alpha > 0$ on a dense subset $G$ on the space $H$ which contains the measurable square integrable functions on $(0, l)$ and secondly the optimal control problem has at least one solution for any $\alpha \geq 0$ on the space $H$.

1. **Introduction.** A great number of theoretical concepts governed by Schrödinger equation [5]-[9] have been examined in the area of optimal control problems. These problems have emerged as a result of the examination process of one dimensional loaded mass' disintegration [3]. Different approaches for these problems have been examined [19]-[10], [1]. The statement of the problem studied in this paper seriously differs from other studies. In this problem, we examined Schrödinger equation with complex coefficient which contains gradient and used Lions functional as cost functional. Optimal control problems with Lions functional have been examined for linear and nonlinear Schrödinger equations with complex coefficient which do not contain gradient in the study of [11]-[16]. As optimal control problems involving gradient have not been studied, this paper plays a significant role both in practical and theoretical aspects.

The methodology of the study follows as: In section 2, we state problem and recall some known spaces. Then generalized solution of the problem is presented and for this solution estimates are given. In section 3, existence and uniqueness theorems particular to solution of this problem are proved. The results are given in section 4.

2010 *Mathematics Subject Classification.* Primary: 49N05, 34A12.
*Key words and phrases.* Optimal control problem, Lions functional, Schrödinger equation.
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2. **Statement of the optimal control problem.** Let $l > 0$, $T > 0$ be given positive numbers, $0 \le x \le l$, $0 \le t \le T$, $\Omega_t = (0,l) \times (0,T)$, $\Omega = \Omega_T$. $C^k([0,T],B)$ is a Banach space the elements of which are the functions defined on $[0,T]$, $k \ge 0$ times continuously differentiable and their values belong to the $B$ Banach space. Here, norm is given as below

$$\|u\|_{C^k([0,T],B)} = \sum_{m=0}^{k} \max_{0\le t\le T} \left\| \frac{d^m u(t)}{dt} \right\|_B \tag{1}$$

For $k \ge 0$ , $m \ge 0$, $W_p^k(0,l)$ and $W_p^{k,m}(\Omega)$ are Sobolev spaces defined as in the study [12].

Our objective is to find out the problem of minimum of the functional

$$J_\alpha(v) = \|\psi_1 - \psi_2\|_{L_2(\Omega)} + \alpha \|v - \omega\|_H \tag{2}$$

on the set $V = \left\{ v = v(x), v \in L_2(0,l), \|v\|_{L_2(0,l)} \le b_0 \right\}$ under the conditions

$$i\frac{\partial \psi_k}{\partial t} + a_0 \frac{\partial^2 \psi_k}{\partial x^2} + i a_1(x) \frac{\partial \psi_k}{\partial x} - a(x)\psi_k + v(x)\psi_k = f_k(x,t), k = 1,2, (x,t) \in \Omega, \tag{3}$$

$$\psi_k(x,0) = \varphi_k(x), k = 1,2, x \in (0,l), \tag{4}$$

$$\psi_1(0,t) = \psi_1(l,t) = 0, \forall t \in (0,T), \tag{5}$$

$$\frac{\partial \psi_2(0,t)}{\partial x} = \frac{\partial \psi_2(l,t)}{\partial x} = 0, t \in (0,T), \tag{6}$$

where $i^2 = -1$, $b_0 > 0$,$\alpha \ge 0$,$a_0 > 0$ are given numbers, $a(x)$,$a_1(x)$ are bounded, measurable functions with real value and they satisfy

$$0 < \mu_0 \le a(x) \le \mu_1, \overset{0}{\forall} x \in (0,l), \mu_0, \mu_1 > 0 \text{ ,are constants} \tag{7}$$

$$|a_1(x)| \le \mu_2, \left| \frac{da_1(x)}{dx} \right| \le \mu_3, \overset{0}{\forall} x \in (0,l), \tag{8}$$

$$a_1(0) = a_1(l) = 0, \mu_2, \mu_3 > 0 \text{ are constants} \tag{9}$$

$\varphi_k(x), f_k(x,t)$, $k = 1,2$ are measurable functions with complex value and they satisfy

$$\varphi_1 \in \overset{0}{W_2^2}(0,l), \varphi_2 \in W_2^2(0,l), \frac{d\varphi_2(0)}{dx} = \frac{d\varphi_2(l)}{dx} = 0, \tag{10}$$

$$f_k \in W_2^{0,1}(\Omega), k = 1,2. \tag{11}$$

Here, the functional is defined as Lions functional. For the first time, Lions type functionals were presented by French mathematician Lions for parabolic equation in optimal control problem [14]. This type functionals have been firstly presented and analyzed by Iskenderov in equations of mathematical physics [10]. $\alpha > 0$ is the regularization parameter [17]. For Schrödinger equation, problem of finding the function$\psi_1 = \psi_1(x,t) = \psi_1(x,t;v)$ from the conditions (3)-(5) is the first kind of initial boundary value problem; finding the function $\psi_2 = \psi_2(x,t) = \psi_2(x,t;v)$ from the conditions (3),(4),(6) is the second kind of initial boundary value problem.

The solution of the problem (3)-(6) clearly relates to the control $v$, therefore we also use $\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v)$.

**Definition 2.1.** $\psi_k = \psi_k(x,t) = \psi_k(x,t;v)$ k=1,2 functions are called the solution of initial boundary value problem (3)-(6) for each $v \in V$, where $\psi_1 \in \overset{0}{W_2^{2,1}}(\Omega)$, $\psi_2 \in W_2^{2,1}(\Omega)$ and they satisfy the conditions (3)-(6) for $\overset{0}{\forall}(x,t) \in \Omega$.

If we consider initial boundary value problem (3)-(6) then by means of the results on solvability of initial boundary value problem for Schrödinger equation, known from the paper [18], we can formulate the following statement:

**Theorem 2.2.** *Assume that $a(x)$,$a_1(x)$, $\varphi_k(x)$, $f_k(x,t)$functions satisfy the conditions (7)-(11). Then the problem (3)-(6) has a unique solution as $\psi_1 \in \overset{0}{W}{}_2^{2,1}(\Omega)$, $\psi_2 \in W_2^{2,1}(\Omega)$ for each $v \in V$ and this solution satisfies subsequent estimates:*

$$\|\psi_1\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega)} \le c_1 \left( \|\varphi_1\|^2_{\overset{0}{W}{}_2^{2}(0,l)} + \|f_1\|^2_{W_2^{0,1}} \right), \tag{12}$$

$$\|\psi_2\|^2_{W_2^{2,1}(\Omega)} \le c_2(\|\varphi_2\|^2_{W_2^{2}(0,l)} + \|f_2\|^2_{W_2^{0,1}(\Omega)}). \tag{13}$$

*Here, the constants numbers $c_1 > 0$, $c_2 > 0$ are independent from $\varphi_k$, $f_k$, $k = 1, 2$.*

3. **Existence and uniqueness of solution of the optimal control problem.** Now, existence and uniqueness of the solution for (2)-(6) optimal control problem will be researched. Primarily, we will prove uniqueness of the solution for $\alpha > 0$. The following theorem can be written from [4].

**Theorem 3.1.** *Let us accept that $\tilde{X}$ is a uniformly convex space, $U$ is a closed, bounded set of $\tilde{X}$ space, $I(v)$ is a lower bounded and lower semi continuous functional defined on $U$ and $\alpha > 0$, $\beta \ge 1$ are given numbers. Then there is such an almost dense subset $G$ in $\tilde{X}$ space that the functional*

$$J_\alpha(v) = \mathrm{I}(v) + \alpha \|v - \omega\|^\beta_{\tilde{X}} \tag{14}$$

*has its minimum value on the set of $U$ for $\forall \omega \in G$. If $\beta > 1$, the functional $J_\alpha(v)$ has its minimum value at a unique point on $U$ [4].*

**Theorem 3.2.** *Assume that the conditions (7)-(11) hold and theorem1 be fulfilled also. Then there is such an almost dense subset $G$ in $H = L_2(0,l)$ that (2)-(6) optimal control problem has a unique solution for $\forall \omega \in G$,$\alpha > 0$.*

*Proof.* Firstly we will prove the continuity of functional

$$J_0(v) = \|\psi_1 - \psi_2\|^2_{L_2(\Omega)} \tag{15}$$

on the set of $V$. Let $\Delta v \in H$ be an increment on $\forall v \in V$element such that $v + \Delta v \in V$. Then the functions$\psi_k = \psi_k(x,t) = \psi_k(x,t;v)$, $k = 1,2$ have $\Delta\psi_k = \psi_k(x,t;v+\Delta v) - \psi_k(x,t;v)$, k=1,2 increments. Here,$\psi_k = \psi_k(x,t) \equiv \psi_k(x,t;v)$, k=1,2 and $\psi_{\Delta k} = \psi_{\Delta k}(x,t) \equiv \psi_k(x,t;v+\Delta v)$, k=1,2 functions are the solutions of (3)-(6) for $v \in V$ and $v + \Delta v \in V$ respectively. From the conditions (3)-(6), it is clear that the functions $\Delta\psi_k = \Delta\psi_k(x,t)$, $k = 1,2$ are the solutions of the following initial boundary value problem:

$$i\frac{\partial \Delta\psi_k}{\partial t} + a_0\frac{\partial^2 \Delta\psi_k}{\partial x^2} + ia_1(x)\frac{\partial \Delta\psi_k}{\partial x} - a(x)\Delta\psi_k + (v(x) + \Delta v(x))\Delta\psi_k$$
$$= -\Delta v(x)\psi_k(x,t;v),\ k = 1,2,\ (x,t) \in \Omega, \tag{16}$$

$$\Delta\psi_k(x,0) = 0, k = 1,2, x \in (0,l), \tag{17}$$

$$\Delta\psi_1(0,t) = \Delta\psi_1(l,t) = 0, t \in (0,T) \tag{18}$$

$$\frac{\partial \Delta\psi_2(0,t)}{\partial x} = \frac{\partial \Delta\psi_2(l,t)}{\partial x} = 0, t \in (0,T). \tag{19}$$

Let us obtain an estimate for the solution of this problem. To this end, by multiplying both sides of equation (3) with $\Delta\overline{\psi}(x,t)$ and integrating over $\Omega_T$, we get following equality:

$$\int_{\Omega_t}\left(i\frac{\partial\Delta\psi_k}{\partial t}\Delta\bar{\psi}_k+a_0\frac{\partial^2\Delta\psi_k}{\partial x^2}\Delta\bar{\psi}_k+ia_1(x)\frac{\partial\Delta\psi_k}{\partial x}\Delta\bar{\psi}_k-a(x)\left|\Delta\psi_k\right|^2\right)dxd\tau$$
$$+\int_{\Omega_t}\left(\left(v(x)+\Delta v(x)\right)\left|\Delta\psi_k\right|^2\right)dxd\tau$$
$$=-\int_{\Omega_t}\Delta v(x)\psi_k(x,\tau)\Delta\psi_k(x,\tau)dxd\tau,\ \forall t\in[0,T],\ k=1,2. \tag{20}$$

In the last equality, if we apply partial integration to the second term in the left side by using (5),(6) boundary conditions, the following equality is acquired:

$$\int_{\Omega_t}\left(i\frac{\partial\Delta\psi_k}{\partial t}\Delta\bar{\psi}_k-a_0\left|\frac{\partial\Delta\psi_k}{\partial x}\right|+ia_1(x)\frac{\partial\Delta\psi_k}{\partial x}\Delta\bar{\psi}_k-a(x)\left|\Delta\psi_k\right|^2\right)dxd\tau$$
$$+\int_{\Omega_t}\left(\left(v(x)+\Delta v(x)\right)\left|\Delta\psi_k\right|^2\right)dxd\tau$$
$$=-\int_{\Omega_t}\Delta v(x)\psi_k(x,\tau)\Delta\psi_k(x,\tau)dxd\tau,\forall t\in[0,T],k=1,2. \tag{21}$$

When the complex conjugate of the above equality is subtracted from itself, the following equality is obtained:

$$\int_{\Omega_t}\left(\frac{\partial\Delta\psi_k}{\partial t}\Delta\bar{\psi}_k+\frac{\partial\Delta\bar{\psi}_k}{\partial t}\Delta\psi_k\right)dxd\tau$$
$$+\int_{\Omega_t}a_1(x)\left(\frac{\partial\Delta\psi_k}{\partial x}\Delta\bar{\psi}_k+\frac{\partial\Delta\bar{\psi}_k}{\partial x}\Delta\psi_k\right)dxd\tau$$
$$=2\int_{\Omega_t}Im\left(\psi_k(x,\tau)\Delta\bar{\psi}_k(x,\tau)\right)\Delta v(x)dxd\tau,\forall t\in[0,T],k=1,2. \tag{22}$$

Let us add the term of $\int_{\Omega_t}\frac{da_1(x)}{dx}\left|\Delta\psi_k\right|^2dxd\tau$ on and subtract it from the both sides of this equality. Then we get following equality:

$$\begin{aligned}&\textstyle\int_{\Omega_t}\left(\frac{\partial\Delta\psi_k}{\partial t}\Delta\bar{\psi}_k+\frac{\partial\Delta\bar{\psi}_k}{\partial t}\Delta\psi_k\right)dxd\tau+\int_{\Omega_t}\frac{\partial}{\partial x}\left(a_1(x)\left|\Delta\psi_k\right|^2\right)dxd\tau\\&=\textstyle\int_{\Omega_t}\frac{da_1(x)}{dx}\left|\Delta\psi_k\right|^2dxd\tau+2\int_{\Omega_t}Im\psi_k(x,\tau)\Delta\bar{\psi}_k(x,\tau)\Delta v(x)dxd\tau,\end{aligned} \tag{23}$$

$\forall t\in[0,T],k=1,2$. Second term in the left side of this equality converts to zero under the conditions $\Delta\psi_1(0,t)=\Delta\psi_1(l,t)$, $a_1(0)=a_1(l)=0$. Considering this fact and using the conditions $\Delta\psi_k(x,0)=0$, $k=1,2$, we can write the last equality as follows:

$$\int_0^l\left|\Delta\psi_k(x,t)\right|^2dx=\int_{\Omega_t}\frac{da_1(x)}{dx}\left|\Delta\psi_k\right|^2dxd\tau$$
$$+2\int_{\Omega_t}Im\psi_k(x,\tau)\Delta\bar{\psi}_k(x,\tau)\Delta v(x)dxd\tau, \tag{24}$$

$\forall t\in[0,T]$, $k=1,2$.

When the condition (9) is used for $a_1(x)$ and the Cauchy- Bunjakovskii inequality is employed, following inequality is obtained:

$$\left\|\Delta\psi_k(.,t)\right\|^2_{L_2(0,l)}\le(1+\mu_3)\int_0^t\left\|\Delta\psi_k(.,\tau)\right\|^2_{L_2(0,l)}d\tau$$

$$+\int_{\Omega_t}\left(|\Delta v(x)|\,|\psi_k(x,\tau)|\right)^2 dxd\tau,\quad \mathrm{k}=1,2. \tag{25}$$

Let us take the second term of the right side of inequality (25). Since $\Delta v \in L_2(0,l)$,the term can be written as follows:

$$\int_{\Omega_t}\left(|\Delta v(x)|\,|\psi_k(x,\tau)|\right)^2 dx \le \|\Delta v\|^2_{L_2(0,l)}\int_0^T \max_{x\in(0,l)}|\psi_k(x,\tau)|^2\,d\tau,\ \forall t\in[0,T]. \tag{26}$$

Since $\psi_1 \in \overset{0}{W}{}_2^{2,1}(\Omega)$ and $\psi_2 \in W_2^{2,1}(\Omega)$, according to Embedding theorem $\psi_1 \in L_2(0,T;\overset{0}{W}{}_2^1(0,l))$ and $\psi_2 \in L_2(0,T;W_2^1(0,l))$ relations are valid. The spaces $\overset{0}{W}{}_2^{2,1}(0,l)$ and $W_2^1(0,l)$ are embedded into $C[0,l]$. So, we can write following inequalities:

$$\|\psi_1\|^2_{L_2(0,T;C(0,l))} \le c_3\,\|\psi_1\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega)}, \tag{27}$$

$$\|\psi_2\|^2_{L_2(0,T;C(0,l))} \le c_4\,\|\psi_2\|^2_{W_2^{2,1}(\Omega)} \tag{28}$$

where the constants $c_3>0$, $c_4>0$ are independent from $\psi_1$, $\psi_2$, respectively. Using these inequalities, the estimates (12), (13) and the inequality (26), we obtain following inequality:

$$\int_{\Omega_t}\left(|\Delta v(x)|\,|\psi_k(x,\tau)|\right)^2 dxd\tau \le c_5\,\|\Delta v\|^2_{L_2(0,l)} \tag{29}$$

$\forall t\in[0,T]$, $k=1,2$.

If we take into account this inequality in (25), we obtain:

$$\|\Delta\psi_k(.,t)\|^2_{L_2(0,l)} \le (1+\mu_3)\int_0^t \|\Delta\psi_k(.,\tau)\|^2_{L_2(0,l)}\,d\tau + c_6\,\|\Delta v\|^2_{L_2(0,l)}, \tag{30}$$

$\forall t\in[0,T]$, $\mathrm{k}=1,2$. By applying Gronwall lemma, we get following estimate:

$$\|\Delta\psi_k(.,t)\|^2_{L_2(0,l)} \le c_7\,\|\Delta v\|^2_{L_2(0,l)} \tag{31}$$

$\forall t\in[0,T]$ , $k=1,2$. If (15) is used to evaluate increment of the functional $J_0(v)$ for$\forall v\in V$, we obtain increment of the functional as shown below:

$$\begin{aligned}\Delta J_0(v) &= \Delta J_0(v+\Delta v) - J_0(v)\\ &= 2\int_\Omega Re\left[(\psi_1(x,t)-\psi_2(x,t))\left(\Delta\bar\psi_1(x,t)-\Delta\bar\psi_2(x,t)\right)\right]dxdt\\ &+\|\Delta\psi_1\|^2_{L_2(\Omega)}+\|\Delta\psi_2\|^2_{L_2(\Omega)}-2\int_\Omega Re\left(\Delta\psi_1(x,t)\Delta\bar\psi_2(x,t)\right)dxd\tau.\end{aligned} \tag{32}$$

Here applying the Cauchy-Bunjakovskii inequality and using the estimates (12), (13) in (32), we obtain following inequality:

$$\|\Delta J_0(v)\| \le c_8\left(\|\Delta\psi_1\|_{L_2(\Omega)}+\|\Delta\psi_2\|_{L_2(\Omega)}+\|\Delta\psi_1\|^2_{L_2(\Omega)}+\|\Delta\psi_2\|^2_{L_2(\Omega)}\right). \tag{33}$$

If we use the estimate (31) in the last inequality, we obtain following relation:

$$\|\Delta J_0(v)\| \le c_9\left(\|\Delta v\|_{L_2(0,l)}+\|\Delta v\|^2_{L_2(0,l)}\right) \tag{34}$$

where $c_9>0$ is independent from $\Delta v$. From the last relation, it is observed that the functional is continuous for $\forall v\in V$. Namely $|\Delta J_0(v)|\to 0$, for$\|\Delta v\|_{L_2(0,l)}\to 0$. Since (34) inequality is valid for $\forall v\in V$, the functional $J_0(v)$ is continuous on the set $V$. On the other hand, $J_0(v)\ge 0$ condition is satisfied for $\forall v\in V$. Namely, the functional $J_0(v)$ is lower bounded on the set $V$. Since the set $V$ is closed, bounded

and convex on the space $L_2(0,l)$and the space $H = L_2(0,l)$ is a smooth convex space [16], all the conditions of Theorem 2 hold. Thus, in accordance with this theorem, there is a dense set $G \subset H$, such that (2)-(6) optimal control problem has a unique solution for every $\omega \in G$ , $\alpha > 0$. Theorem 3 is proved. □

The next theorem points out (2)-(6) optimal control problem has at least one solution for $\alpha \geq 0$ , $\omega \in H$.

**Theorem 3.3.** *Assume that $\alpha \geq 0$ and the hypotheses of Theorem 3 hold. Then (2)-(6) problem has at least one solution for $\forall \omega \in H$.*

*Proof.* Let $\{v^m\} \subset V$ be a minimizing sequence for $J_\alpha(v)$ functional and

$$\lim_{m\to\infty} J_\alpha(v^m) = J_{\alpha *} = \inf_{v\in V} J_\alpha(v). \tag{35}$$

Suppose that $\psi_k^m(x,t) = \psi_k(x,t;v^m)$ k=1,2, m=1,2, ... Since $\forall v^m \in V$, if we consider what we said above the existence and uniqueness' solution of the initial boundary value problem (3)-(6), it can be said that problem has $\psi_k^m(x,t)$ k=1,2 solutions for each m=1,2, ..., furthermore (36)-(37) estimates are valid for these solutions:

$$\|\psi_1^m\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega)} \leq c_{10}\left(\|\varphi_1\|^2_{\overset{0}{W}{}_2^{2}(0,l)} + \|f_1\|^2_{W_2^{0,1}(\Omega)}\right) = c_{11}, \tag{36}$$

$$\|\psi_2^m\|^2_{W_2^{2,1}(\Omega)} \leq c_{12}\left(\|\varphi_2\|^2_{W_2^{2}(0,l)} + \|f_2\|^2_{W_2^{0,1}(\Omega)}\right) = c_{13}, m = 1,2,\ldots \tag{37}$$

where $c_{11} > 0$,$c_{13} > 0$ are independent from m. Since$L_2(0,l)$is a reflexive Banach space and the set $V$ is closed, bounded and convex on the space$L_2(0,l)$,the set $V$ is weakly closed and weakly compact on the space $L_2(0,l)$. Therefore, we can select a subsequence of $\{v^m\}$ which converges to $v \in V$. Let us denote this subsequence as $\{v^m\}$ for simplicity. Thus, following limit relation is obtained:

$$\lim_{m\to\infty} \int_0^l v^m(x)q(x)dx = \int_0^l v(x)q(x)dx \tag{38}$$

for $\forall q \in L_2(0,l)$. From the estimates (36) and (37), the sequences $\{\psi_1^m(x,t)\}$, $\{\psi_2^m(x,t)\}$ are smooth bounded in $\overset{0}{W}{}_2^{2,1}$, $W_2^{2,1}(\Omega)$, respectively. So we can choose subsequences of $\{\psi_1^m(x,t)\}$ and $\{\psi_2^m(x,t)\}$which weakly converge respectively $\psi_1(x,t)$ and $\psi_2(x,t)$ functions in $W_2^{2,1}(\Omega)$space. For simplicity, we can denote this subsequence as $\{\psi_1^m(x,t)\}$, $\{\psi_2^m(x,t)\}$. Finally, we write following limit relations:

$$\psi_k^m \to \psi_k \quad , \quad \text{weakly in } L_2(\Omega \tag{39}$$

$$\frac{\partial \psi_k^m}{\partial x} \to \frac{\partial \psi_k}{\partial x} \quad , \quad \text{weakly in } L_2(\Omega) \tag{40}$$

$$\frac{\partial^2 \psi_k^m}{\partial x^2} \to \frac{\partial^2 \psi_k}{\partial x^2} \quad , \quad \text{weakly in } L_2(\Omega) \tag{41}$$

$$\frac{\partial \psi_k^m}{\partial t} \to \frac{\partial \psi_k}{\partial t} \quad , \quad \text{weakly in } L_2(\Omega) \tag{42}$$

for $m \to \infty$. Now we will show the limit functions $\psi_k(x,t)$, $k = 1,2$ are the solutions of the initial boundary value problem (3)-(6). Firstly, we will demonstrate that $\psi_k(x,t)$, k=1,2 satisfy (2) for $\overset{0}{\forall}(x,t) \in \Omega$. If we use (39)-(42), we can write following limit relation. When $\forall \eta \in L_2(\Omega)$, for $m \to \infty$:

$$\int_\Omega \left(i\frac{\partial \psi_k^m}{\partial t} + a_0\frac{\partial^2 \psi_k^m}{\partial x^2} + ia_1(x)\frac{\partial \psi_k^m}{\partial x} - a(x)\psi_k^m\right)\bar{\eta}dxdt$$

$$\rightarrow \int_{\Omega}\left(i\frac{\partial\psi_k}{\partial t}+a_0\frac{\partial^2\psi_k}{\partial x^2}+ia_1(x)\frac{\partial\psi_k}{\partial x}-a(x)\psi_k\right)\bar{\eta}dxdt,\ \mathrm{k}=1,2\ . \tag{43}$$

Now let us prove that following limit relation is valid when $\forall\eta\in L_2(\Omega)$, for $m\rightarrow\infty$:

$$\int_{\Omega}v^m(x)\psi_k^m(x,t)\bar{\eta}(x,t)dxdt\rightarrow\int_{\Omega}v(x)\psi_k(x,t)\bar{\eta}(x,t)dxdt,\ \mathrm{k}=1,2. \tag{44}$$

It is clear that (45) equality is valid:

$$\int_{\Omega}v^m(x)\psi_k^m(x,t)\bar{\eta}(x,t)dxdt=\int_{\Omega}(v^m(x)-v(x))\psi_k(x,t)\bar{\eta}dxdt$$
$$+\int_{\Omega}v^m(x)\left(\psi_k^m(x,t)-\psi_k(x,t)\right)\bar{\eta}(x,t)dxdt+\int_{\Omega}v(x)\psi_k(x,t)\bar{\eta}(x,t)dxdt,\ \mathrm{k}=1,2\ . \tag{45}$$

Using the (38) limit relation and considering $q(x)=\int_0^T\psi_k(x,t)\bar{\eta}(x,t)dt$ function is an element of $L_2(0,l)$ for $\psi_k(x,t)\eta(x,t)$functions, which satisfy $\psi_k\in W_2^{2,1}(\Omega)$, $k=1,2$, $\eta\in L_2(\Omega)$, we can say that limit of the first term of the right side of equality (45) is equal to zero, for $m\rightarrow\infty$, namely

$$\lim_{m\rightarrow\infty}\int_{\Omega}\left(v^m(x)-v(x)\right)\psi_k(x,t)\bar{\eta}(x,t)dxdt=0 \tag{46}$$

$k=1,2$.

Since $\overset{0}{W}{}_2^{2,1}(\Omega)$, $W_2^{2,1}(\Omega)$ spaces are compactly embedded into $L_2(0,T;L_\infty(0,l))$, which is known from [17], we can write following limit relation:

$$\|\psi_k^m-\psi_k\|_{L_2(0,T;L_\infty(0,l))}\rightarrow 0,k=1,2 \tag{47}$$

for $m\rightarrow\infty$. Let us find limit of the second term in the right side of (45) by using the last limit relation. If we take into account this term, we get following inequality:

$$\begin{array}{l}\left|\int_{\Omega}v^m(x)\left(\psi_k^m(x,t)-\psi_k(x,t)\right)\bar{\eta}(x,t)dxdt\right|\\ \leq\|v^m\|_{L_2(0,l)}\|\eta\|_{L_2(\Omega)}\|\psi_k^m-\psi_k\|_{L_2(0,T;L_\infty(0,l))}\\ \leq\ b_0\|\eta\|_{L_2(\Omega)}\|\psi_k^m-\psi_k\|_{L_2(0,T;L_\infty(0,l))}\ .\end{array} \tag{48}$$

If we use limit relation (47) and take limit of both sides of (45), we obtain that term in the right side of (45) converges zero, i.e:

$$\lim_{m\rightarrow\infty}\int_{\Omega}v^m(x)\left(\psi_k^m(x,t)-\psi_k(x,t)\right)\bar{\eta}(x,t)dxdt=0 \tag{49}$$

$k=1,2$. Hence using the limit relations (46), (49) and taking limit of both sides of (45) for $m\rightarrow\infty$, we obtain that (44) is valid. Eventually, using the limit relations (43),(44) and taking limit of

$$\int_{\Omega}\left(i\frac{\partial\psi_k^m}{\partial t}+a_0\frac{\partial^2\psi_k^m}{\partial x^2}+ia_1(x)\frac{\partial\psi_k^m}{\partial x}-a(x)\psi_k^m+v^m(x)\psi_k^m-f_k(x,t)\right)\times\bar{\eta}(x,t)dxdt \tag{50}$$

is 0, $k=1,2$. We obtain that $\psi_k(x,t)$, k=1,2 functions satisfy (3).

Now let us prove that $\psi_k(x,t)$, k=1,2 functions satisfy the initial conditions (4). Since the spaces $\overset{0}{W}{}_2^{2,1}(\Omega)$ and $W_2^{2,1}(\Omega)$ are compactly embedded into the space $C^0\left([0,T],L_2(0,l)\right)$, following limit relations can be written:

$$\|\psi_k^m(x,t)-\psi_k(x,t)\|_{L_2(0,l)}\rightarrow 0,k=1,2,\ \text{for}\ m\rightarrow\infty \tag{51}$$

If we use this limit relation for t=0 and initial condition$\psi_{\mathrm{k}}^{\mathrm{m}}(x,0)=\varphi_k(x)$, k=1,2, x$\in(0,l)$ and if we take limit of

$$\|\psi_k(.,0)-\varphi_k\|_{L_2(0,l)}\leq\|\psi_k(.,0)-\psi_k^m(.,0)\|_{L_2(0,l)}+\|\psi_k^m(.,0)-\varphi_k\|_{L_2(0,l)}\,k=1,2 \tag{52}$$

for $m\to\infty$, we get $\|\psi_k(.,0)-\varphi_k\|_{L_2(0,l)}=0$. So, we obtain initial conditions $\psi_k(x,0)=\varphi_k(x)$, k=1,2 , $\overset{0}{\forall}x\in(0,l)$. We will prove that $\psi_k(x,t)$ limit functions satisfy the boundary conditions (5),(6). Firstly, take into account (5). According to the theorem which is about the trace of functions, $\psi_1^m\in\overset{0}{W}{}_2^{2,1}(\Omega)$ functions have traces which are in $L_2(0,T)$ and following limit relations are valid for $m\to\infty$:

$$\|\psi_1^m(s,.)-\psi_1(s,.)\|_{L_2(0,T)}\to 0,\,s=0,l. \tag{53}$$

If we use this limit relations, the boundary conditions$\psi_1^m(0,t)=\psi_1^m(l,t)=0$, $t\in(0,T)$ and take limit of the both sides of

$$\|\psi_1(s,.)\|_{L_2(0,T)}\leq\|\psi_1(s,.)-\psi_1^m(s,.)\|_{L_2(0,T)}+\|\psi_1^m(s,.)\|_{L_2(0,T)}\,, \tag{54}$$

$s=0,l$ for $m\to\infty$, we get relation $\|\psi_1(s,.)\|_{L_2(0,T)}=0$, $s=0,l$ . So, we obtain that the boundary conditions $\psi_1(0,t)=\psi_1(l,t)=0$, $\overset{0}{\forall}t\in(0,T)$ are valid. Now let us prove that the condition (6) is satisfied.

Since $\psi_2^m(x,t)$, m=1,2... functions are elements of $W_2^{2,1}(\Omega)$ , according to Embedding theorem, following limit relations

$$\frac{\partial\psi_2^m(s,t)}{\partial x}\to\frac{\partial\psi_2(s,t)}{\partial x},\,s=0,l,\text{weakly in }L_2(0,T) \tag{55}$$

are valid for $m\to\infty$. Namely

$$\lim_{m\to\infty}\int_0^T\frac{\partial\psi_2^m(s,t)}{\partial x}\bar{\eta}(t)dt=\int_0^T\frac{\partial\psi_2(s,t)}{\partial x}\bar{\eta}(t)dt,\,s=0,l\text{ for}\forall\eta\in L_2(0,T). \tag{56}$$

If we use this limit relation and take into account the boundary conditions

$$\frac{\partial\psi_2^m(0,t)}{\partial x}=\frac{\partial\psi_2^m(l,t)}{\partial x}=0 \tag{57}$$

$t\in(0,T)$, m=1,2,... and take limit of the both sides of

$$\int_0^T\frac{\partial\psi_2(s,t)}{\partial x}\bar{\eta}(t)dt=\int_0^T\left(\frac{\partial\psi_2(s,t)}{\partial x}-\frac{\partial\psi_2^m(s,t)}{\partial x}\right)\bar{\eta}(t)dt+\int_0^T\frac{\partial\psi_2^m(s,t)}{\partial x}dt,\,s=0, \tag{58}$$

we obtain equality $\int_0^T\frac{\partial\psi_2(s,t)}{\partial x}\bar{\eta}(t)dt=0$ for $\forall\eta\in L_2(0,T)$. Hence, we get $\frac{\partial\psi_2(s,t)}{\partial x}=0$, $\overset{0}{\forall}t\in(0,T)$, $s=0,l$. Namely $\psi_2(x,t)$limit function satisfies the boundary condition (6). So, we obtain that $\psi_k(x,t)$, k=1,2 functions are solutions of the (2)-(6) problem corresponding to $v=v(x)\in V$which is the limit function of the sequence $\{v^m\}\subset V$and these functions are from the spaces $\overset{0}{W}{}_2^{2,1}(\Omega)$ and $W_2^{2,1}(\Omega)$, respectively. Namely,

$$\psi_k=\psi_k(x,t)=\psi_k(x,t;v),\ \mathrm{k}=1,2. \tag{59}$$

That (12), (13) estimates are valid for these functions is obtained through taking limit of (36), (37) and considering norm is weakly lower semi continuous in the spaces $\overset{0}{W}{}_2^{2,1}(\Omega)$, $W_2^{2,1}(\Omega)$.

Since $\{v^m\} \subset V$minimizing subsequence and $\{\psi_k^m(x,t)\}\, k = 1,2$ sequences converge weakly to $\psi_k(x,t)$, k=1,2 respectively in $H = L_2(0,l)$and $L_2(\Omega)$, $\alpha \geq 0$ and norm functions of the spaces $L_2(\Omega)$, $H = L_2(0,l)$ are weakly lower semi continuous, we obtain that the functional $J_\alpha(v)$ is weakly lower semi continuous for $v \in V$, i.e

$$J_{\alpha *} \leq J_\alpha(v) \leq \underline{\lim}_{m\to\infty} J_\alpha(v^m) = J_{\alpha *} \tag{60}$$

So, $J_\alpha(v) = J_{\alpha *}$ is obtained. Namely,$v \in V$is the element which minimizes$J_\alpha(v)$. In other words $v \in V$ is the solution of (2)-(6) optimal control problem. Theorem 4 is proved. □

4. **Conclusion.** As a result, we obtain the existence and uniqueness of the solution of an optimal control problem for a Schrödinger equation. Because of the difference of the considered linear Schrödinger equation and conditions, this study is different from previous studies in the literature. As the different from the others, we examined a one- dimensional linear Schrödinger equation which contains a gradient term with complex coefficient. Consequently the goal has been successfully achieved

**Acknowledgment.** This study has been formed out of Gökçe Dilek Küçük's master's thesis which is numbered as 294880 at National Thesis Center of The Council of Higher Education (YÖK) of Turkey.

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Received March 2017; revised July 2017.

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