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A Note on the New Ostrowski and Hadamard Type Inequalities via the Hölder–İşcan Inequality

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Abstract: For all convex functions, the Hermite–Hadamard inequality is already well known in convex analysis. In this regard, Hermite–Hadamard and Ostrowski type inequalities were obtained using exponential type convex functions in this work. In addition, new generalizations were found for different values of θ. In conclusion, we believe that our work's technique will inspire more study in this field.

Keywords: exponential type convex functions; Hölder Inequality; Hölder–İşcan Inequality

MSC: 26A15; 26A51; 26D10

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1. Introduction

The Hermite–Hadamard inequality, which is a major result of the widespread use and great geometrical interpretation of convex functions, has attracted a lot of attention in fundamental mathematics. Recent years have seen remarkable advancement in the inequality field ([1–4]). One of the most significant causes of this development is significant inequalities, such as the Hermite–Hadamard inequality. It is interesting considering how closely connected the theories of inequality and convexity are to one another. The theory of novel convexity has received various new extensions, refinements, and definitions in recent years, and corresponding advancements in the theory of convexity inequality, especially integral inequalities theory, have also been addressed.

The definition of the convex function can be represented as follows:

Definition 1. *A function $\vartheta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\vartheta(\phi\varkappa_1 + (1-\phi)\varkappa_2) \leq \phi\vartheta(\varkappa_1) + (1-\phi)\vartheta(\varkappa_2)$$

holds for all $\varkappa_1, \varkappa_2 \in I$ and $\phi \in [0,1]$.

The Hermite–Hadamard inequality is formally expressed as follows:

We let $\vartheta : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$.

$$\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi \leq \frac{\vartheta(\varkappa_1)+\vartheta(\varkappa_2)}{2}. \tag{1}$$

The inequality in (1) holds in reverse directions if ϑ is a concave function. The geometry-based Hermite–Hadamard inequality provides an upper and lower estimate for the integral mean of any convex function defined on a closed and limited domain, which includes the function's endpoints and midpoint. Due to the significance of this inequality, several

variations of the Hermite–Hadamard inequality have been examined in the literature for various classes of convexity, including harmonically convex, exponentially convex, s-convex, h-convex, and co-ordinate convex functions ([5–9]).

In 1938, the classical integral inequality was established by Ostrowski as follows:

Theorem 1. *Let $\vartheta : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a differentiable mapping on $(\varkappa_1, \varkappa_2)$ whose derivative $\vartheta' : (\varkappa_1, \varkappa_2) \to \mathbb{R}$ is bounded on $(\varkappa_1, \varkappa_2)$, i.e., $\|\vartheta'\|_\infty = \sup_{\phi \in (\varkappa_1, \varkappa_2)} |\vartheta'(\phi)| < \infty$. Then, the following inequality is obtained:*

$$\left|\vartheta(\zeta) - \frac{1}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \vartheta(\phi)d\phi\right| \leq \left[\frac{1}{4} + \frac{\left(\zeta - \frac{\varkappa_1+\varkappa_2}{2}\right)^2}{(\varkappa_2 - \varkappa_1)^2}\right](\varkappa_2 - \varkappa_1)\|\vartheta'\|_\infty$$

for all $\zeta \in [\varkappa_1, \varkappa_2]$. The constant $\frac{1}{4}$ is the best possible.

In the theory of inequality, convex functions play a fundamental role. Various definitions of convex functions have been the subject of investigations by numerous authors. Previous studies have focused on convexity types such as quasi-convex, p-convex, m-convex, (α, m)-convex and especially s-convex. However, recent studies have found that many new types of convexity have been obtained. The exponential type convex functions is one of these new forms of convexity. We provide the following new definition:

Definition 2 ([10]). *A nonnegative function $\vartheta : I \to \mathbb{R}$ is called exponential type convex function if, for every $\varkappa_1, \varkappa_2 \in I$ and $\phi \in [0, 1]$,*

$$\vartheta(\phi\varkappa_1 + (1-\phi)\varkappa_2) \leq (e^\phi - 1)\vartheta(\varkappa_1) + (e^{1-\phi} - 1)\vartheta(\varkappa_2).$$

The class of all exponential type convex functions on interval I is demonstrated by $EXPC(I)$.

In [10], the exponential type convex functions has been described by Kadakal and İşcan, and additional inequalities based on this definition have been established. With this definition, researchers have defined different types of convexities (such as exponential trigonometric convex, exponentially convex, exponentially tgs-convex, exponentially $(\alpha, h-m)$-convex, exponentially (α, m)-convex in [11–19]) and obtained new inequalities:

Remark 1 ([10]). *The range of the exponential type convex functions is $[0, \infty)$.*

Lemma 1 ([10]). *For all $\phi \in [0, 1]$, inequalities $e^\phi - 1 \geq \phi$ and $e^{1-\phi} - 1 \geq 1 - \phi$ hold.*

Proposition 1 ([10]). *Every nonnegative convex function is exponential type convex function.*

Also in [10], using this definition, Kadakal and İşcan obtained new integral inequalities as follows:

Theorem 2. *Let $\vartheta : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be an exponential type convex function. If $\varkappa_1 < \varkappa_2$ and $\vartheta \in L[\varkappa_1, \varkappa_2]$, then the following Hermite–Hadamard type inequalities holds:*

$$\frac{1}{2(\sqrt{e}-1)}\vartheta\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1}\int_{\varkappa_1}^{\varkappa_2} \vartheta(\phi)d\phi \leq (e-2)[\vartheta(\varkappa_1) + \vartheta(\varkappa_2)].$$

Theorem 3. *Let $\vartheta : I \to \mathbb{R}$ be a differentiable function on I°(interior of I), $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$, and assume that $\vartheta' \in L[\varkappa_1, \varkappa_2]$. If $|\vartheta'|$ is an exponential type convex function on $[\varkappa_1, \varkappa_2]$; then, the following is obtained:*

$$\begin{aligned} &\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\ \leq\ & (\varkappa_2-\varkappa_1)\left(4\sqrt{e}-e-\frac{7}{2}\right)\left[\frac{|\vartheta'(\varkappa_1)|+|\vartheta'(\varkappa_2)|}{2}\right]. \end{aligned} \tag{2}$$

Theorem 4. *Let $\vartheta : I \to \mathbb{R}$ be a differentiable function on I°(interior of I), $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$, $q > 1$ and assume that $\vartheta' \in L[\varkappa_1, \varkappa_2]$. If $|\vartheta'|^q$ is an exponential type convex function on $[\varkappa_1, \varkappa_2]$; then, the following is obtained:*

$$\begin{aligned} &\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\ \leq\ & \left(\frac{\varkappa_2-\varkappa_1}{2}\right)[2(e-2)]^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|\vartheta'(\varkappa_1)|^q+|\vartheta'(\varkappa_2)|^q}{2}\right]^{\frac{1}{q}}. \end{aligned} \tag{3}$$

Theorem 5. *Let $\vartheta : I \to \mathbb{R}$ be a differentiable function on I°(interior of I), $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$, $q \geq 1$ and assume that $\vartheta' \in L[\varkappa_1, \varkappa_2]$. If $|\vartheta'|^q$ is an exponential type convex function on $[\varkappa_1, \varkappa_2]$; then, the following is obtained:*

$$\begin{aligned} &\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\ \leq\ & \frac{\varkappa_2-\varkappa_1}{2^{2-\frac{1}{q}}}\left[2(4\sqrt{e}-e-\frac{7}{2})\right]^{\frac{1}{q}}\left[\frac{|\vartheta'(\varkappa_1)|^q+|\vartheta'(\varkappa_2)|^q}{2}\right]^{\frac{1}{q}}. \end{aligned} \tag{4}$$

Hölder inequality is the basis of many theorems and inequalities. Therefore, the authors have widely used the Hölder inequality and the power-mean inequality as a result of Hölder inequality in their research.

In [20], İşcan obtained the Hölder–İşcan inequality and showed that this inequality produces better upper bounds than the Hölder inequality. By using the Hölder–İşcan inequality, many new results were obtained in the literature (see [20–25]).

The Hölder–İşcan inequality is defined as follows in [20]:

Theorem 6. *Let $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If ϑ and $\hbar$ are real functions defined on $[\varkappa_1, \varkappa_2]$ and if $|\vartheta|^p$ and $|\hbar|^q$ are integrable on $[\varkappa_1, \varkappa_2]$; then,*

$$\begin{aligned} \int_{\varkappa_1}^{\varkappa_2}|\vartheta(\phi)\hbar(\phi)|d\phi \ \leq\ & \frac{1}{\varkappa_2-\varkappa_1}\left\{\left(\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\phi)|\vartheta(\phi)|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-\phi)|\hbar(\phi)|^q d\phi\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\varkappa_1}^{\varkappa_2}(\phi-\varkappa_1)|\vartheta(\phi)|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\varkappa_2}(\phi-\varkappa_1)|\hbar(\phi)|^q d\phi\right)^{\frac{1}{q}}\right\} \\ \leq\ & \left(\int_{\varkappa_1}^{\varkappa_2}|\vartheta(\phi)|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\varkappa_2}|\hbar(\phi)|^q d\phi\right)^{\frac{1}{q}}. \end{aligned}$$

The main purpose of this paper is to obtain new inequalities for exponential convex functions by using Identity (5). In line with this purpose, we achieve a new integral identity for continuously differentiable functions. This identity helps as an auxiliary result to obtain the main results of the article. Also, we obtain different types of inequalities according to different values of θ.

2. Main Results

First of all, we need to establish the following new lemma, which plays an important role in obtaining the main results of the article:

Lemma 2 ([26]). *Let $\vartheta : I \to \mathbb{R}$ be a differentiable function on I°(interior of I), $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$, $\zeta \in [\varkappa_1, \varkappa_2]$ and assume that $\vartheta' \in L[\varkappa_1, \varkappa_2]$ (Integrable in the Lebesgue means). Then, the following is obtained:*

$$\begin{aligned} & \int_{\varkappa_1}^{\varkappa_2} K(\zeta,\phi)\vartheta'(\phi)d\phi \\ = \;& (1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi, \end{aligned} \tag{5}$$

where

$$K(\zeta,\phi)=\begin{cases} \phi-[\theta\zeta+(1-\theta)\varkappa_1], & \phi\in[\varkappa_1,\zeta], \\ \phi-[\theta\zeta+(1-\theta)\varkappa_2], & \phi\in(\zeta,\varkappa_2]. \end{cases}$$

Theorem 7. *Let $\vartheta : I \to \mathbb{R}$ be a differentiable function on I°, $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$, $q > 1$ and assume that $\vartheta' \in L[\varkappa_1, \varkappa_2]$. If $|\vartheta'|^q$ is an exponential type convex function on $[\varkappa_1, \varkappa_2]$; then, inequality*

$$\begin{aligned} & \left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\ \leq \;& (\zeta-\varkappa_1)^{\frac{p+1}{p}}(\varkappa_2-\varkappa_1)^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\ & \times\left\{(\zeta+e-\varkappa_1-e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\varkappa_1-\zeta-1+e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}} \\ & +(\varkappa_2-\zeta)^{\frac{p+1}{p}}(\varkappa_2-\varkappa_1)^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\ & \times\left\{(\varkappa_2-\zeta-1+e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\zeta+e-\varkappa_2-e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}} \end{aligned}$$

holds for $\varkappa_1 \leq \zeta \leq \varkappa_2$, $0 \leq \theta \leq 1$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using identity (5) and Hölder's inequality, we have

$$\begin{aligned} & \left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\ \leq \;& \int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|\left|\vartheta'\left(\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}\varkappa_1+\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}\varkappa_2\right)\right|d\phi \\ & +\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|\left|\vartheta'\left(\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}\varkappa_1+\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}\varkappa_2\right)\right|d\phi \\ \leq \;& \left(\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\zeta}\left|\vartheta'\left(\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}\varkappa_1+\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}\varkappa_2\right)\right|^q d\phi\right)^{\frac{1}{q}} \\ & +\left(\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\zeta}^{\varkappa_2}\left|\vartheta'\left(\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}\varkappa_1+\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}\varkappa_2\right)\right|^q d\phi\right)^{\frac{1}{q}}. \end{aligned}$$

Therefore, $|\vartheta'|^q$ is an exponential type convex function on $[\varkappa_1, \varkappa_2]$. We obtain

$$\begin{aligned}
&\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ &\left(\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\zeta}\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi\right)^{\frac{1}{q}} \\
&+\left(\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|^p d\phi\right)^{\frac{1}{p}}\left(\int_{\zeta}^{\varkappa_2}\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi\right)^{\frac{1}{q}} \\
=\ &(\zeta-\varkappa_1)^{\frac{p+1}{p}}(\varkappa_2-\varkappa_1)^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\
&\times\left\{(\zeta+e-\varkappa_1-e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\varkappa_1-\zeta-1+e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}} \\
&+(\varkappa_2-\zeta)^{\frac{p+1}{p}}(\varkappa_2-\varkappa_1)^{\frac{1}{q}}\left(\frac{\theta^{p+1}+(1-\theta)^{p+1}}{p+1}\right)^{\frac{1}{p}} \\
&\times\left\{(\varkappa_2-\zeta-1+e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\zeta+e-\varkappa_2-e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

The result obtained below is related to the left-hand side of the Ostrowski inequality:

Corollary 1. *If we choose $\theta = 0$ in Theorem 7, we obtain*

$$\begin{aligned}
&\left|\vartheta(\zeta)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ &\left(\frac{1}{(p+1)(\varkappa_2-\varkappa_1)}\right)^{\frac{1}{p}} \\
&\times\left[(\zeta-\varkappa_1)^{\frac{p+1}{p}}\left\{(\zeta+e-\varkappa_1-e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\varkappa_1-\zeta-1+e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right. \\
&\left.+(\varkappa_2-\zeta)^{\frac{p+1}{p}}\left\{(\varkappa_2-\zeta-1+e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\zeta+e-\varkappa_2-e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2. *If we choose $\zeta = \frac{\varkappa_1+\varkappa_2}{2}$ in Corollary 1, we have the following inequality that is related to the left-hand side of Hadamard integral inequalities;*

$$\begin{aligned}
&\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ &\frac{\varkappa_2-\varkappa_1}{2}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}}\left\{\left[\left(e-\sqrt{e}+\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_1)\right|^q+\left(\sqrt{e}-1-\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_2)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\sqrt{e}-1+\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_1)\right|^q+\left(e-\sqrt{e}-\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_2)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

The result obtained below is a generalized version of the Hermite–Hadamard inequality.

Corollary 3. *If we choose $\theta = 1$ in Theorem 7, we obtain*

$$\begin{aligned}
&\left|\frac{(\zeta-\varkappa_1)\vartheta(\varkappa_1)+(\varkappa_2-\zeta)\vartheta(\varkappa_2)}{\varkappa_2-\varkappa_1}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \left(\frac{1}{(p+1)(\varkappa_2-\varkappa_1)}\right)^{\frac{1}{p}} \\
&\times\left[(\zeta-\varkappa_1)^{\frac{p+1}{p}}\left\{(\zeta+e-\varkappa_1-e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\varkappa_1-\zeta-1+e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right. \\
&\left.+\ (\varkappa_2-\zeta)^{\frac{p+1}{p}}\left\{(\varkappa_2-\zeta-1+e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_1)\right|^q+(\zeta+e-\varkappa_2-e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}})\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 4. *If we choose $\zeta=\frac{\varkappa_1+\varkappa_2}{2}$ in Corollary 3, we have the following inequality that is related to the right-hand side of Hadamard integral inequalities for exponential-type convex functions;*

$$\begin{aligned}
&\left|\frac{\vartheta(\varkappa_1)+\vartheta(\varkappa_2)}{2}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{\varkappa_2-\varkappa_1}{2}\left(\frac{1}{2p+2}\right)^{\frac{1}{p}} \\
&\times\left\{\left[\left(e-\sqrt{e}+\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_1)\right|^q+\left(\sqrt{e}-1-\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_2)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\sqrt{e}-1+\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_1)\right|^q+\left(e-\sqrt{e}-\frac{\varkappa_2-\varkappa_1}{2}\right)\left|\vartheta'(\varkappa_2)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 8. *Let $\vartheta: I\to\mathbb{R}$ be a differentiable function on I°, $\varkappa_1,\varkappa_2\in I^\circ$ with $\varkappa_1<\varkappa_2$, $q>1$ and assume that $\vartheta'\in L[\varkappa_1,\varkappa_2]$. If $|\vartheta'|^q$ is an exponential type convex function on $[\varkappa_1,\varkappa_2]$; then, we the following inequality is obtained:*

$$\begin{aligned}
&\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \qquad (6)\\
\leq\ & \left(\frac{(\zeta-\varkappa_1)^2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{\left[\theta^{p+1}(p+2-\theta)+(1-\theta)^{p+2}\right]^{\frac{1}{p}}\varphi^{\frac{1}{q}}\right. \\
&\left.+\left[(1-\theta)^{p+1}(p+1+\theta)+\theta^{p+2}\right]^{\frac{1}{p}}\chi^{\frac{1}{q}}\right\} \\
&+\left(\frac{(\varkappa_2-\zeta)^2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{\left[\theta^{p+1}(p+2-\theta)+(1-\theta)^{p+2}\right]^{\frac{1}{p}}\mu^{\frac{1}{q}}\right. \\
&\left.+\left[(1-\theta)^{p+1}(p+1+\theta)+\theta^{p+2}\right]^{\frac{1}{p}}\omega^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
\varphi &= \left|\vartheta'(\varkappa_1)\right|^q\left((\varkappa_2-\varkappa_1)^2e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)(\zeta-\varkappa_2)e-\frac{(\zeta-\varkappa_1)^2}{2}\right)\\
&\quad+\left|\vartheta'(\varkappa_2)\right|^q\left((\varkappa_2-\varkappa_1)^2e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)(2\varkappa_1-\varkappa_2-\zeta)-\frac{(\zeta-\varkappa_1)^2}{2}\right),\\
\chi &= \left|\vartheta'(\varkappa_1)\right|^q\left((\varkappa_2-\varkappa_1)(2\varkappa_1-\varkappa_2-\zeta)e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)^2e-\frac{(\zeta-\varkappa_1)^2}{2}\right)\\
&\quad+\left|\vartheta'(\varkappa_2)\right|^q\left((\varkappa_2-\varkappa_1)(\zeta-\varkappa_2)e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)^2-\frac{(\zeta-\varkappa_1)^2}{2}\right),\\
\mu &= \left|\vartheta'(\varkappa_1)\right|^q\left((\varkappa_2-\varkappa_1)^2e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)(\varkappa_1-2\varkappa_2+\zeta)-\frac{(\varkappa_2-\zeta)^2}{2}\right)\\
&\quad+\left|\vartheta'(\varkappa_2)\right|^q\left((\varkappa_2-\varkappa_1)^2e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)(\varkappa_1-\zeta)e-\frac{(\varkappa_2-\zeta)^2}{2}\right),\\
\omega &= \left|\vartheta'(\varkappa_1)\right|^q\left((\varkappa_2-\varkappa_1)(\varkappa_1-\zeta)e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)^2-\frac{(\varkappa_2-\zeta)^2}{2}\right)\\
&\quad+\left|\vartheta'(\varkappa_2)\right|^q\left((\varkappa_2-\varkappa_1)(\varkappa_1-2\varkappa_2+\zeta)e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}+(\varkappa_2-\varkappa_1)^2e-\frac{(\varkappa_2-\zeta)^2}{2}\right)
\end{aligned}$$

and $\varkappa_1\le\zeta\le\varkappa_2$, $0\le\theta\le1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Identity (5) and Hölder–İşcan integral inequality, we obtain

$$\begin{aligned}
&\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right|\\
\le&\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|\left|\vartheta'(\phi)\right|d\phi+\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|\left|\vartheta'(\phi)\right|d\phi\\
\le&\frac{1}{\zeta-\varkappa_1}\left\{\left(\int_{\varkappa_1}^{\zeta}(\zeta-\phi)|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|^pd\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\zeta}(\zeta-\phi)\left|\vartheta'(\phi)\right|^qd\phi\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_{\varkappa_1}^{\zeta}(\phi-\varkappa_1)|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|^pd\phi\right)^{\frac{1}{p}}\left(\int_{\varkappa_1}^{\zeta}(\phi-\varkappa_1)\left|\vartheta'(\phi)\right|^qd\phi\right)^{\frac{1}{q}}\right\}\\
&+\frac{1}{\varkappa_2-\zeta}\left\{\left(\int_{\zeta}^{\varkappa_2}(\varkappa_2-\phi)|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|^pd\phi\right)^{\frac{1}{p}}\left(\int_{\zeta}^{\varkappa_2}(\varkappa_2-\phi)\left|\vartheta'(\phi)\right|^qd\phi\right)^{\frac{1}{q}}\right.\\
&\qquad\left.+\left(\int_{\zeta}^{\varkappa_2}(\phi-\zeta)|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|^pd\phi\right)^{\frac{1}{p}}\left(\int_{\zeta}^{\varkappa_2}(\phi-\zeta)\left|\vartheta'(\phi)\right|^qd\phi\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By the exponential type convexity of $|\vartheta'|^q$, we have

$$\begin{aligned}
&\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{1}{\zeta-\varkappa_1}\Bigg\{\Bigg(\int_{\varkappa_1}^{\theta\zeta+(1-\theta)\varkappa_1}(\zeta-\phi)([\theta\zeta+(1-\theta)\varkappa_1]-\phi)^p d\phi \\
&+\int_{\theta\zeta+(1-\theta)\varkappa_1}^{\zeta}(\zeta-\phi)(\phi-[\theta\zeta+(1-\theta)\varkappa_1])^p d\phi\Bigg)^{\frac{1}{p}} \\
&\times\left(\int_{\varkappa_1}^{\zeta}(\zeta-\phi)\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi\right)^{\frac{1}{q}} \\
&+\Bigg(\int_{\varkappa_1}^{\theta\zeta+(1-\theta)\varkappa_1}(\phi-\varkappa_1)([\theta\zeta+(1-\theta)\varkappa_1]-\phi)^p d\phi \\
&+\int_{\theta\zeta+(1-\theta)\varkappa_1}^{\zeta}(\phi-\varkappa_1)(\phi-[\theta\zeta+(1-\theta)\varkappa_1])^p d\phi\Bigg)^{\frac{1}{p}} \\
&\times\left(\int_{\varkappa_1}^{\zeta}(\phi-\varkappa_1)\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi\right)^{\frac{1}{q}}\Bigg\} \\
&+\frac{1}{\varkappa_2-\zeta}\Bigg\{\Bigg(\int_{\zeta}^{\theta\zeta+(1-\theta)\varkappa_2}(\varkappa_2-\phi)([\theta\zeta+(1-\theta)\varkappa_2]-\phi)^p d\phi \\
&+\int_{\theta\zeta+(1-\theta)\varkappa_2}^{\varkappa_2}(\varkappa_2-\phi)(\phi-[\theta\zeta+(1-\theta)\varkappa_2])^p d\phi\Bigg)^{\frac{1}{p}} \\
&\times\left(\int_{\zeta}^{\varkappa_2}(\varkappa_2-\phi)\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi\right)^{\frac{1}{q}} \\
&\Bigg(+\int_{\zeta}^{\theta\zeta+(1-\theta)\varkappa_2}(\phi\text{-}\zeta)([\theta\zeta+(1-\theta)\varkappa_2]-\phi)^p d\phi \\
&+\int_{\theta\zeta+(1-\theta)\varkappa_2}^{\varkappa_2}(\phi-\zeta)(\phi-[\theta\zeta+(1-\theta)\varkappa_2])^p d\phi\Bigg)^{\frac{1}{p}} \\
&\times\int_{\zeta}^{\varkappa_2}(\phi-\zeta)\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi\Bigg\}.
\end{aligned}$$

If the necessary calculations are performed in the inequality given above, the desired result (6) can be obtained. □

Corollary 5. *If we choose $\theta = 0$ in Theorem 8, we obtain*

$$\begin{aligned}
&\left|\vartheta(\zeta)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{1}{\varkappa_2-\varkappa_1}\Bigg[\left(\frac{(\zeta-\varkappa_1)^2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{\varphi^{\frac{1}{q}}+(p+1)^{\frac{1}{p}}\chi^{\frac{1}{q}}\right\} \\
&+\left(\frac{(\varkappa_2-\zeta)^2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{\mu^{\frac{1}{q}}+(p+1)^{\frac{1}{p}}\omega^{\frac{1}{q}}\right\}\Bigg].
\end{aligned}$$

φ, χ, μ and ω are as in Theorem 8.

Corollary 6. *If we choose* $\zeta = \frac{\varkappa_1+\varkappa_2}{2}$ *in Corollary 5, we obtain the following inequality that is related to the left-hand side of Hadamard integral inequalities:*

$$\begin{aligned}
&\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{\varkappa_2-\varkappa_1}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}} \\
&\times\Bigg\{\left[\left|\vartheta'(\varkappa_1)\right|^q(8\sqrt{e}-4e-1)+\left|\vartheta'(\varkappa_2)\right|^q(8\sqrt{e}-13)\right]^{\frac{1}{q}} \\
&+(p+1)^{\frac{1}{p}}\left[\left|\vartheta'(\varkappa_1)\right|^q(12\sqrt{e}+8e-1)+\left|\vartheta'(\varkappa_2)\right|^q(4\sqrt{e}+7)\right]^{\frac{1}{q}} \\
&+(p+1)^{\frac{1}{p}}\left[\left|\vartheta'(\varkappa_1)\right|^q(-4\sqrt{e}+7)+\left|\vartheta'(\varkappa_2)\right|^q(-12\sqrt{e}+8e-1)\right]^{\frac{1}{q}} \\
&+\left[\left|\vartheta'(\varkappa_1)\right|^q(8\sqrt{e}-13)+\left|\vartheta'(\varkappa_2)\right|^q(8\sqrt{e}-4e-1)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Corollary 7. *If we choose* $\theta = 1$ *in Theorem 8, we obtain*

$$\begin{aligned}
&\left|\frac{(\zeta-\varkappa_1)\vartheta(\varkappa_1)+(\varkappa_2-\zeta)\vartheta(\varkappa_2)}{\varkappa_2-\varkappa_1}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{1}{\varkappa_2-\varkappa_1}\left[\left(\frac{(\zeta-\varkappa_1)^2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{(p+1)^{\frac{1}{p}}\varphi^{\frac{1}{q}}+\chi^{\frac{1}{q}}\right\}\right. \\
&\left.+\left(\frac{(\varkappa_2-\zeta)^2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{(p+1)^{\frac{1}{p}}\mu^{\frac{1}{q}}+\omega^{\frac{1}{q}}\right\}\right].
\end{aligned}$$

φ, χ, μ and ω are as in Theorem 8.

Corollary 8. *If we choose* $\zeta = \frac{\varkappa_1+\varkappa_2}{2}$ *in Corollary 7, we obtain the following inequality that is related to the right-hand side of Hadamard integral inequalities for exponential type convex functions:*

$$\begin{aligned}
&\left|\frac{\vartheta(\varkappa_1)+\vartheta(\varkappa_2)}{2}-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{\varkappa_2-\varkappa_1}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}} \\
&\times\Bigg\{(p+1)^{\frac{1}{p}}\left[\left|\vartheta'(\varkappa_1)\right|^q(8\sqrt{e}-4e-1)+\left|\vartheta'(\varkappa_2)\right|^q(8\sqrt{e}-13)\right]^{\frac{1}{q}} \\
&+\left[\left|\vartheta'(\varkappa_1)\right|^q(12\sqrt{e}+8e-1)+\left|\vartheta'(\varkappa_2)\right|^q(4\sqrt{e}+7)\right]^{\frac{1}{q}} \\
&+\left[\left|\vartheta'(\varkappa_1)\right|^q(-4\sqrt{e}+7)+\left|\vartheta'(\varkappa_2)\right|^q(-12\sqrt{e}+8e-1)\right]^{\frac{1}{q}} \\
&+(p+1)^{\frac{1}{p}}\left[\left|\vartheta'(\varkappa_1)\right|^q(8\sqrt{e}-13)+\left|\vartheta'(\varkappa_2)\right|^q(8\sqrt{e}-4e-1)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Theorem 9. *Let $\vartheta : I \to \mathbb{R}$ be a differentiable function on I°, $\varkappa_1, \varkappa_2 \in I^\circ$ with $\varkappa_1 < \varkappa_2$, $q \geq 1$ and assume that $\vartheta' \in L[\varkappa_1, \varkappa_2]$. If $|\vartheta'|^q$ is an exponential type convex function on $[\varkappa_1, \varkappa_2]$, then the following inequality can be obtained:*

$$\begin{aligned} &\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \quad (7)\\ \leq\ &\left[(\theta\zeta+(1-\theta)\varkappa_1)((\theta-1)\zeta-\theta\varkappa_1)+\frac{\zeta^2+\varkappa_1^2}{2}\right]^{1-\frac{1}{q}}(\Psi_1)^{\frac{1}{q}}\\ &+\left[(\theta\zeta+(1-\theta)\varkappa_2)((\theta-1)\zeta-\theta\varkappa_2)+\frac{\zeta^2+\varkappa_2^2}{2}\right]^{1-\frac{1}{q}}(\Psi_2)^{\frac{1}{q}}, \end{aligned}$$

where $\varkappa_1 \leq \zeta \leq \varkappa_2$, $0 \leq \theta \leq 1$ and

$$\begin{aligned} \Psi_1 = &\left|\vartheta'(\varkappa_1)\right|^q\Bigg[2(\varkappa_2-\varkappa_1)^2e^{\frac{(\varkappa_2-\varkappa_1)-(\zeta-\varkappa_1)\theta}{\varkappa_2-\varkappa_1}}\\ &+(\varkappa_2-\varkappa_1)\left[e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}(\varkappa_1-\varkappa_2+(1-\theta)(\varkappa_1-\zeta))+e(\varkappa_1-\varkappa_2+(\zeta-\varkappa_1)\theta)\right]\\ &+(\theta\zeta+(1-\theta)\varkappa_1)(\theta\varkappa_1+(1-\theta)\zeta)-\frac{\zeta^2+\varkappa_1^2}{2}\Bigg]\\ &+\left|\vartheta'(\varkappa_2)\right|^q\Bigg[2(\varkappa_2-\varkappa_1)^2e^{\frac{(\zeta-\varkappa_1)\theta}{\varkappa_2-\varkappa_1}}\\ &+(\varkappa_2-\varkappa_1)\left[e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}(\zeta-\varkappa_2+(\zeta-\varkappa_1)\theta)+(\varkappa_1-\varkappa_2+(\varkappa_1-\zeta)\theta)\right]\\ &+(\theta\zeta+(1-\theta)\varkappa_1)(\theta\varkappa_1+(1-\theta)\zeta)-\frac{\zeta^2+\varkappa_1^2}{2}\Bigg], \end{aligned}$$

$$\begin{aligned} \Psi_2 = &\left|\vartheta'(\varkappa_1)\right|^q\Bigg[2(\varkappa_2-\varkappa_1)^2e^{\frac{(\varkappa_2-\zeta)\theta}{\varkappa_2-\varkappa_1}}\\ &+(\varkappa_2-\varkappa_1)\left[e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}(\varkappa_1-\zeta+(\varkappa_2-\zeta)\theta)+(\varkappa_1-\varkappa_2+(\zeta-\varkappa_2)\theta)\right]\\ &+(\theta\zeta+(1-\theta)\varkappa_2)(\theta\varkappa_2+(1-\theta)\zeta)-\frac{\zeta^2+\varkappa_2^2}{2}\Bigg]\\ &+\left|\vartheta'(\varkappa_2)\right|^q\Bigg[2(\varkappa_2-\varkappa_1)^2e^{\frac{(\varkappa_2-\varkappa_1)-(\varkappa_2-\zeta)\theta}{\varkappa_2-\varkappa_1}}\\ &+(\varkappa_2-\varkappa_1)\left[e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}(\varkappa_1-\varkappa_2+(1-\theta)(\zeta-\varkappa_2))+e(\varkappa_1-\varkappa_2+(\varkappa_2-\zeta)\theta)\right]\\ &+(\theta\zeta+(1-\theta)\varkappa_2)(\theta\varkappa_2+(1-\theta)\zeta)-\frac{\zeta^2+\varkappa_2^2}{2}\Bigg]. \end{aligned}$$

Proof. We assume that $q > 1$. By using Identity (5), power-mean inequality and the property of the exponential type convex of $|\vartheta'|^q$, we obtain

$$\begin{aligned}
&\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|\left|\vartheta'(\phi)\right|d\phi+\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|\left|\vartheta'(\phi)\right|d\phi \\
\leq\ & \left(\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|d\phi\right)^{1-\frac{1}{q}}\left(\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|\left|\vartheta'(\phi)\right|^q d\phi\right)^{\frac{1}{q}} \\
&+\left(\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|d\phi\right)^{1-\frac{1}{q}}\left(\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|\left|\vartheta'(\phi)\right|^q d\phi\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|d\phi \\
=\ & \int_{\varkappa_1}^{\theta\zeta+(1-\theta)\varkappa_1}([\theta\zeta+(1-\theta)\varkappa_1]-\phi)d\phi+\int_{\theta\zeta+(1-\theta)\varkappa_1}^{\zeta}(\phi-[\theta\zeta+(1-\theta)\varkappa_1])d\phi \\
=\ & (\theta\zeta+(1-\theta)\varkappa_1)((\theta-1)\zeta-\theta\varkappa_1)+\frac{\zeta^2+\varkappa_1^2}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|d\phi \\
=\ & \int_{\zeta}^{\theta\zeta+(1-\theta)\varkappa_2}([\theta\zeta+(1-\theta)\varkappa_2]-\phi)d\phi+\int_{\theta\zeta+(1-\theta)\varkappa_2}^{\varkappa_2}(\phi-[\theta\zeta+(1-\theta)\varkappa_2])d\phi \\
=\ & (\theta\zeta+(1-\theta)\varkappa_2)((\theta-1)\zeta-\theta\varkappa_2)+\frac{\zeta^2+\varkappa_2^2}{2}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_{\varkappa_1}^{\zeta}|\phi-[\theta\zeta+(1-\theta)\varkappa_1]|\left|\vartheta'(\phi)\right|^q d\phi \\
\leq\ & \int_{\varkappa_1}^{\theta\zeta+(1-\theta)\varkappa_1}([\theta\zeta+(1-\theta)\varkappa_1]-\phi)\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi \\
&+\int_{\theta\zeta+(1-\theta)\varkappa_1}^{\zeta}(\phi-[\theta\zeta+(1-\theta)\varkappa_1])\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi \\
=\ & \Psi_1,
\end{aligned}$$

$$\begin{aligned}
&\int_{\zeta}^{\varkappa_2}|\phi-[\theta\zeta+(1-\theta)\varkappa_2]|\left|\vartheta'(\phi)\right|^q d\phi \\
\leq\ & \int_{\zeta}^{\theta\zeta+(1-\theta)\varkappa_2}([\theta\zeta+(1-\theta)\varkappa_2]-\phi)\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi \\
&+\int_{\theta\zeta+(1-\theta)\varkappa_2}^{\varkappa_2}(\phi-[\theta\zeta+(1-\theta)\varkappa_2])\left[(e^{\frac{\varkappa_2-\phi}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_1)\right|^q+(e^{\frac{\phi-\varkappa_1}{\varkappa_2-\varkappa_1}}-1)\left|\vartheta'(\varkappa_2)\right|^q\right]d\phi \\
=\ & \Psi_2.
\end{aligned}$$

Thus, the proof is completed. □

Corollary 9. *Under the assumptions of Theorem 9, by choosing $q = 1$, we obtain*

$$\begin{aligned} &\left|(1-\theta)(\varkappa_2-\varkappa_1)\vartheta(\zeta)+\theta(\zeta-\varkappa_1)\vartheta(\varkappa_1)+\theta(\varkappa_2-\zeta)\vartheta(\varkappa_2)-\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \quad (8)\\ \leq\ & I(\varkappa_2,\zeta,\theta)\left|\vartheta'(\varkappa_1)\right|+I(\zeta,\varkappa_1,\theta)\left|\vartheta'(\varkappa_2)\right|, \end{aligned}$$

where $\varkappa_1 \leq \zeta \leq \varkappa_2$, $0 \leq \theta \leq 1$ and

$$\begin{aligned} &I(\varkappa_2,\zeta,\theta)\\ =\ & 2(\varkappa_2-\varkappa_1)^2(e^{1-\frac{\theta(\zeta-\varkappa_1)}{\varkappa_2-\varkappa_1}}+e^{\frac{\theta(\varkappa_2-\zeta)}{\varkappa_2-\varkappa_1}})\\ &+(\varkappa_2-\varkappa_1)\left\{e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}}[(1-\theta)(\varkappa_1+\varkappa_2-2\zeta)+2(\varkappa_1-\varkappa_2)]+\theta\zeta(e+1)\right.\\ &\quad\left.+(1-\theta)(e\varkappa_1+\varkappa_2)+(\varkappa_1-2\varkappa_2-e\varkappa_2)\right\}\\ &+(\theta\zeta+(1-\theta)\varkappa_1)(\theta\varkappa_1+(1-\theta)\zeta)+(\theta\zeta+(1-\theta)\varkappa_2)(\theta\varkappa_2+(1-\theta)\zeta)-\frac{2\zeta^2+\varkappa_1^2+\varkappa_2^2}{2}, \end{aligned}$$

$$\begin{aligned} &I(\zeta,\varkappa_1,\theta)\\ =\ & 2(\varkappa_2-\varkappa_1)^2(e^{1-\frac{\theta(\varkappa_2-\zeta)}{\varkappa_2-\varkappa_1}}+e^{\frac{\theta(\zeta-\varkappa_1)}{\varkappa_2-\varkappa_1}})\\ &+(\varkappa_2-\varkappa_1)\left\{e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}}[(1-\theta)(-\varkappa_1-\varkappa_2+2\zeta)+2(\varkappa_1-\varkappa_2)]+\theta\zeta(-e-1)\right.\\ &\quad\left.+(1-\theta)(-e\varkappa_2-\varkappa_1)+(2\varkappa_1-\varkappa_2+e\varkappa_1)\right\}\\ &+(\theta\zeta+(1-\theta)\varkappa_1)(\theta\varkappa_1+(1-\theta)\zeta)+(\theta\zeta+(1-\theta)\varkappa_2)(\theta\varkappa_2+(1-\theta)\zeta)-\frac{2\zeta^2+\varkappa_1^2+\varkappa_2^2}{2}. \end{aligned}$$

Remark 2. *If we choose $\zeta = \frac{\varkappa_1+\varkappa_2}{2}$ in Inequality (8), we obtain the following inequality that is related to the left-hand side of Hadamard integral inequalities:*

$$\begin{aligned} &\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right)-\frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \quad (9)\\ \leq\ & (\varkappa_2-\varkappa_1)\left(\frac{3}{4}+e-2\sqrt{e}\right)\left\{\left|\vartheta'(\varkappa_1)\right|+\left|\vartheta'(\varkappa_2)\right|\right\}. \end{aligned}$$

Inequality (9) is better than Inequality (2), because, for $e \leq 2{,}72$ and $\sqrt{e} \leq 1{,}64$, if we use Inequality (9), we obtain

$$\begin{aligned} &(\varkappa_2-\varkappa_1)\left(\frac{3}{4}+e-2\sqrt{e}\right)\left\{\left|\vartheta'(\varkappa_1)\right|+\left|\vartheta'(\varkappa_2)\right|\right\}\\ =\ & (\varkappa_2-\varkappa_1)\left(\frac{3}{2}+2e-4\sqrt{e}\right)\left\{\frac{|\vartheta'(\varkappa_1)|+|\vartheta'(\varkappa_2)|}{2}\right\}\\ \leq\ & \frac{356(\varkappa_2-\varkappa_1)}{1000}\left\{\frac{|\vartheta'(\varkappa_1)|+|\vartheta'(\varkappa_2)|}{2}\right\}. \end{aligned}$$

Also, if we use Inequality (2), we can write

$$\begin{aligned} &(\varkappa_2-\varkappa_1)\left(4\sqrt{e}-e-\frac{7}{2}\right)\left[\frac{|\vartheta'(\varkappa_1)|+|\vartheta'(\varkappa_2)|}{2}\right]\\ \leq\ & \frac{364(\varkappa_2-\varkappa_1)}{1000}\left\{\frac{|\vartheta'(\varkappa_1)|+|\vartheta'(\varkappa_2)|}{2}\right\}. \end{aligned}$$

Therefore, Inequality (9) is better than Inequality (2).

The following new result was obtained as an Ostrowski type inequality:

Corollary 10. *If we choose* $\theta = 0$ *in Theorem 9, we obatin*

$$\begin{aligned}
&\left|\vartheta(\zeta) - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \left(\frac{(\zeta-\varkappa_1)^2}{2(\varkappa_2-\varkappa_1)}\right)^{1-\frac{1}{q}}\left\{\left[(\varkappa_2-\varkappa_1)e + (2a-\varkappa_2-\zeta)e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}} - \frac{(\zeta-\varkappa_1)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_1)\right|^q\right. \\
&\left.+\left[(\varkappa_2-\varkappa_1) + (\zeta-\varkappa_2)e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}} - \frac{(\zeta-\varkappa_1)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}} \\
&+\left(\frac{(\varkappa_2-\zeta)^2}{2(\varkappa_2-\varkappa_1)}\right)^{1-\frac{1}{q}}\left\{\left[(\varkappa_2-\varkappa_1) + (\varkappa_1-\zeta)e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}} - \frac{(\varkappa_2-\zeta)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_1)\right|^q\right. \\
&\left.+\left[(\varkappa_2-\varkappa_1)e + (\varkappa_1-2b+\zeta)e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}} - \frac{(\varkappa_2-\zeta)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Corollary 11. *If we choose* $\zeta = \frac{\varkappa_1+\varkappa_2}{2}$ *in Corollary 10, we obtain the following inequality that is related to the left-hand side of Hadamard integral inequalities:*

$$\begin{aligned}
&\left|\vartheta\left(\frac{\varkappa_1+\varkappa_2}{2}\right) - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{\varkappa_2-\varkappa_1}{8}\left[\left\{\left[8e-12\sqrt{e}-1\right]\left|\vartheta'(\varkappa_1)\right|^q + \left[7-4\sqrt{e}\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right. \\
&\left.+\left\{\left[7-4\sqrt{e}\right]\left|\vartheta'(\varkappa_1)\right|^q + \left[8e-12\sqrt{e}-1\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 12. *If we choose* $\theta = 1$ *in Theorem 9, we obtain*

$$\begin{aligned}
&\left|\frac{(\zeta-\varkappa_1)\vartheta(\varkappa_1)+(\varkappa_2-\zeta)\vartheta(\varkappa_2)}{\varkappa_2-\varkappa_1} - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \left(\frac{(\zeta-\varkappa_1)^2}{2(\varkappa_2-\varkappa_1)}\right)^{1-\frac{1}{q}}\left\{\left[(\zeta-\varkappa_2)e + (\varkappa_2-\varkappa_1)e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}} - \frac{(\zeta-\varkappa_1)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_1)\right|^q\right. \\
&\left.+\left[(2a-\varkappa_2-\zeta) + (\varkappa_2-\varkappa_1)e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}} - \frac{(\zeta-\varkappa_1)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}} \\
&+\left(\frac{(\varkappa_2-\zeta)^2}{2(\varkappa_2-\varkappa_1)}\right)^{1-\frac{1}{q}}\left\{\left[(\varkappa_1-2b+\zeta) + (\varkappa_2-\varkappa_1)e^{\frac{\varkappa_2-\zeta}{\varkappa_2-\varkappa_1}} - \frac{(\varkappa_2-\zeta)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_1)\right|^q\right. \\
&\left.+\left[(\varkappa_1-\zeta)e + (\varkappa_2-\varkappa_1)e^{\frac{\zeta-\varkappa_1}{\varkappa_2-\varkappa_1}} - \frac{(\varkappa_2-\zeta)^2}{2(\varkappa_2-\varkappa_1)}\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Corollary 13. *If we choose* $\zeta = \frac{\varkappa_1+\varkappa_2}{2}$ *in Corollary 12, we obtain the following inequality that is related to the right-hand side of Hadamard integral inequalities for exponential type convex functions:*

$$\begin{aligned}
&\left|\frac{\vartheta(\varkappa_1)+\vartheta(\varkappa_2)}{2} - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\vartheta(\phi)d\phi\right| \\
\leq\ & \frac{\varkappa_2-\varkappa_1}{8}\left[\left\{\left[8\sqrt{e}-4e-1\right]\left|\vartheta'(\varkappa_1)\right|^q + \left[8\sqrt{e}-13\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right. \\
&\left.+\left\{\left[8\sqrt{e}-13\right]\left|\vartheta'(\varkappa_1)\right|^q + \left[8\sqrt{e}-4e-1\right]\left|\vartheta'(\varkappa_2)\right|^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

3. Conclusions

In this paper, different types of inequalities (Hermite–Hadamard Inequality, Ostrowski Inequality) were found according to different values of θ by using exponential type convex functions. In addition, researchers can obtain different types of inequalities according to different values of θ. The Hölder–İşcan inequality and several lemmas can be used to produce novel results for various kinds of convexity. We are certain that the novel consequences and approaches discussed in this paper will inspire scholars to look into a more intriguing continuation in this area. On the other hand, we believe that if we use the weighted integrals of [27,28], much more general results can be obtained.

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