Honam Mathematical J. **47** (2025), No. 4, pp. 498–512
https://doi.org/10.5831/HMJ.2025.47.4.498

DARBOUX-FRAMED BCV-LEGENDRE CURVE AND ASSOCIATED GEOMETRY

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Abstract. In this study, the components of the Darboux frame associated with a Legendre curve lying on a surface in the BCV-Sasakian space are derived for the first time. The geometric invariants of the curve, namely its normal curvature, geodesic curvature, and geodesic torsion, are explicitly computed. A canal surface is then constructed around the Darboux-framed BCV-Legendre curve, leading to the formulation of Darboux-framed BCV-Legendre tubes. The intrinsic and extrinsic geometry of these tubular surfaces is investigated by calculating their Gaussian and mean curvatures. Moreover, the relationships between the parameter curves and geodesics, asymptotic curves, and curvature lines on the Darboux-framed BCV-Legendre tubes are analyzed in detail. All of these results are obtained for the first time in the context of BCV-Sasakian geometry, thus providing an original and significant contribution to the differential geometry of contact manifolds.

1. Introduction

While the geometric structures of even-dimensional spaces are typically studied in symplectic and complex manifolds, those of odd-dimensional spaces are studied in contact manifolds. Sasakian manifolds, which are odd-dimensional manifolds, are also called normal contact metric manifolds. Curves are fundamental elements in differential geometry, and studying their properties is key to understanding the shape and behavior of manifolds and various spaces in both theory and practice [2],[1]. Legendre curves, in particular, are of fundamental importance in contact manifolds due to their natural association with the contact structure. In contact manifolds, a diffeomorphism is a contact transformation if and only if every Legendre curve in its domain is mapped to another Legendre curve [3]. Belkhelfa et al. investigated Legendre curves in Riemannian and Lorentzian Sasaki spaces [5]. The Bianchi-Cartan-Vranceanu

Received January 18, 2025. Accepted October 20, 2025.
2020 Mathematics Subject Classification. 53A04, 65D17.
Key words and phrases. Legendre curve, BCV Sasakian manifold, Darboux frame, Tubular surface, Bianchi-Cartan-Vranceanu spaces.
This work was supported by the Scientific and Technological Research Council of Türkiye (TÜBİTAK) under project number 1059B192300644
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(BCV) metric, introduced as a family of Riemannian metrics, has a Sasakian structure on the Euclidean 3-space and this structure is defined as the three-dimensional BCV-Sasakian manifold ($\mathfrak{M}^3$). Yıldırım and Hacısalihoğlu studied the Legendre curves and defined the Frenet frame on $\mathfrak{M}^3$ [18]. However, there is no study in the literature that provides the components of the Darboux frame for a Legendre curve lying on a surface in this space.

The canal surface is defined as the envelope of a variable radius moving sphere [13],[17]. Tube surfaces represent the special case where the radius remains constant. Tube surfaces are used to represent the geometry of three-dimensional tubular objects that arise in practical contexts such as engineering and computer-aided design, and due to their practical importance, they are one of the topics studied in differential geometry. Tube surfaces have been studied in various geometric structures such as the BCV-Sasakian space, the Galilean space, and Euclidean space. Indeed, in Euclidean space, they have been examined using different frame fields, including the Frenet frame, Bishop frame, and Darboux frame. However, to the best of our knowledge, there is no study addressing tube surfaces based on the Darboux frame in the BCV-Sasakian space [11],[10],[12],[15].

The Darboux frame is an alternative approach to constructing a moving frame defined on a surface. The orthonormal frame field $\{T, V, U\}$, defined along a curve lying on a surface—where T is the unit tangent vector of the curve, U is the unit normal vector of the surface, and $V = U \times T$ —is known as the Darboux frame. Although the geometry of surfaces in 3-dimensional homogeneous space has been studied extensively with the Bianchi-Cartan-Vranceanu metric, no study focused on the Darboux frame.

This study makes a novel and significant contribution to the field of differential geometry, specifically within the rich framework of BCV-Sasakian geometry. For the first time, it rigorously derives the components of the Darboux frame associated with a Legendre curve lying on a surface in the BCV-Sasakian space, unveiling new geometric insights into these fundamental structures. By explicitly computing key geometric invariants — including the normal curvature, geodesic curvature, and geodesic torsion — this work deepens our understanding of the intrinsic and extrinsic properties of such curves. Building upon this foundation, the study introduces the construction of Darboux-framed BCV-Legendre canal surfaces and Darboux-framed BCV-Legendre tubes, advancing the theory of tubular surfaces in contact geometry. The investigation of Gaussian and mean curvatures of these tubular surfaces, along with a detailed analysis of parameter curves in relation to geodesics, asymptotic curves, and lines of curvature, provides comprehensive characterizations that have not been previously explored in the BCV-Sasakian context. Collectively, these original results open new avenues for the geometric analysis of contact manifolds, enriching both theoretical knowledge and potential applications.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is called a contact manifold if it admits a global 1-form η

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere in M. In this case, the 1-form η is referred to as a contact structure. It is well known that there exists a unique vector field ξ satisfying $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ on a contact manifold (M, η). This vector field ξ is called the characteristic vector field (or Reeb) associated with η. The contact distribution D associated with η is defined as

$$D = \{X \epsilon \chi(M) : \eta(X) = 0\}.$$

Moreover, there exists an endomorphism field φ of rank $2n$, satisfying the conditions

$$\varphi^2(X) = -X + \eta(X)\xi, \ \eta(\varphi(X)) = 0, \ \varphi(\xi) = 0,$$

such that the quadruple (M, η, ξ, φ) is called an almost contact manifold. If there exists a Riemannian metric g on M satisfying

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y) \\ & \quad g(\xi, X) = \eta(X), \end{aligned}$$

then $(M, \eta, \xi, \varphi, g)$ is called an almost contact metric manifold. Furthermore, if g satisfies

$$g(\varphi X, Y) = -d\eta(X, Y),$$

then the structure is referred to as a contact metric manifold. It is well known that every contact metric manifold is a contact manifold [3]. The necessary and sufficient condition for an almost contact metric manifold $(M, \eta, \xi, \varphi, g)$ to be a Sasakian manifold is that

$$(\nabla_{X\varphi})Y = g(X, Y)\xi - \eta(Y)X,$$

holds for all vector fields X and Y on M, where ∇ is Levi-Civita connection. In addition, if the tensor $N^{(1)} : \chi(M) \times \chi(M) \to \chi(M)$ defined by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi$$

vanishes identically, then the tensor $N^{(1)}$ is called the Sasakian tensor, and the contact manifold $(M, \eta, \xi, \varphi, g)$ is called a Sasakian manifold [3].

Definition 2.1. *Let $(M, \eta, \xi, \varphi, g)$ be a 3-dimensional, almost contact metric manifold. The cross product $\times$ on $T_P M$, the tangent space at a point $P \in M$, is defined by*

$$\begin{aligned} X \times Y &= -g(X, \varphi Y)\xi - \eta(Y)\varphi(X) + \eta(X)\varphi(Y), \\ \|X \times Y\|^2 &= g(X, X)g(Y, Y) - g(X, Y)^2, \end{aligned}$$

for any X, $Y \in T_P M$. This cross product satisfies the following identities:

$$\begin{aligned} Y \times \varphi X &= g(X,Y)\xi - \eta(Y)X, \\ \varphi X &= \xi \times X \end{aligned}$$

(see [8]).

Let $\mathbf{R}^3$ be endowed with the Bianchi-Cartan-Vranceanu (BCV) metric $g_{\lambda,\mu}$ defined as

$$g_{\lambda,\mu} = \frac{dx_1^2 + dx_2^2}{\left[1 + \mu(x_1^2 + x_2^2)\right]^2} + [dx_3 + \frac{\lambda}{2}\frac{x_2 dx_1 - x_1 dx_2}{1 + \mu(x_1^2 + x_2^2)}]^2,$$

where $\lambda, \mu \in \mathbf{R}$ and $1 + \mu(x_1^2 + x_2^2) \neq 0$. The space $(\mathbf{R}^3, g_{\lambda,\mu})$ is called the BCV-space, and is denoted by $\mathfrak{M}^3$ or $\mathfrak{M}^3_{\lambda,\mu}$ [9]. The space $\mathfrak{M}^3$ has dimension three, that is, $dim\mathfrak{M}^3 = 3$. If $\lambda = 0$, $\mu = 0$, then the space $\mathfrak{M}^3$ reduces to the Euclidean space, denoted by E^3. In the special case where $\lambda \neq 0$, $\mu = 0$ the space $\mathfrak{M}^3$ is called Heisenberg space, denoted by N^3 [14]. In 1894, and later in 1928 Bianchi classified Riemannian metrics in the 3-dimensional Euclidean space E^3 [6],[7]. Cartan and Vranceanu published several studies related to these spaces [9],[16].

If η, ξ, φ are defined as

$$\begin{aligned} \eta &= dx_3 + \frac{\lambda}{2}\frac{x_2 dx_1 - x_1 dx_2}{1 + \mu(x_1^2 + x_2^2)}, \\ \xi &= \frac{\partial}{\partial x_3}, \\ \varphi &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

on $\chi(\mathfrak{M}^3)$ for $\lambda \neq 0$, then the following relations hold:

$$\begin{aligned} d\eta(X,Y) &= \frac{\lambda}{2} g_{\lambda,\mu}(X, \varphi(Y)), \\ (\nabla_{X\varphi})Y &= \frac{\lambda}{2}\{g_{\lambda,\mu}(X,Y)\xi - \eta(Y)X\}. \end{aligned} \tag{1}$$

In this case, the structure $(\mathfrak{M}^3, \eta, \xi, \varphi, g_{\lambda,\mu})$ together with the relations above (1), forms a Sasakian manifold, known as the BCV-Sasakian space (See [5],[4],[18],[17] for more details).

Definition 2.2. *The subspace $D_m = \{X \epsilon T_{\mathfrak{M}^3}(m) : \eta(X) = 0\}$ is called the contact distribution. A 1-dimensional integral submanifold of D_m is referred to as a Legendre curve [3]. According to this definition, a Legendre curve on $\mathfrak{M}^3$ can be expressed as*

$$\gamma : I \to D_m \subset \mathfrak{M}^3, \ s \to \gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)),$$

where $m = \gamma(s)$. Since $\mathfrak{M}^3$ is the BCV-Sasakian space, the Legendre curve in $\mathfrak{M}^3$ specifically referred to as a BCV-Legendre curve [18].

Let $\gamma : I \to D_m \subset \mathfrak{M}^3$ be a BCV-Legendre curve parameterized by arc length s in $\mathfrak{M}^3$. Furthermore, for the Frenet vector fields $\{V_1, V_2, V_3\}$ of the Legendre curve, the Frenet equations are given by

$$\begin{bmatrix} \nabla_{V_1} V_1 \\ \nabla_{V_1} V_2 \\ \nabla_{V_1} V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

where κ denotes the curvature, the torsion τ is given by

$$\tau = \frac{\lambda}{2},$$

and

$$V_1 = \dot{\gamma}, V_2 = \varphi V_1, V_3 = \xi$$

(see [18]).

3. Geometric Constructions Based on a BCV-Legendre Curve

In this section, we focus on a set of differential geometric constructions derived from a BCV-Legendre curve in the BCV-Sasakian space. First, we compute the components of the Darboux frame associated with the curve and analyze its geometric properties. Then, using this frame, we construct two types of surfaces: the canal surface equipped with the Darboux frame and the tube surface formed around the given BCV-Legendre curve. Furthermore, we investigate the relationship between the parameter curves and certain special curves—namely geodesics, asymptotic curves, and lines of curvature—on the BCV-Legendre tube. These constructions not only provide insight into the local geometry of the curve and the associated surfaces but also reveal the intricate interplay between curvature, torsion, and the Sasakian structure.

3.1. Construction of the Darboux Frame

Lemma 3.1. *Let $\mathfrak{M} \subset D_m$ be a submanifold in $\mathfrak{M}^3$ and let $\gamma : I \to D_m \subset \mathfrak{M}^3$ be a BCV-Legendre curve parameterized by arc length s on $\mathfrak{M}$. Then, a well-defined Darboux frame $\{V_1, Y, U\}$ along the BCV-Legendre curve γ is obtained from the Frenet frame of γ via the following relation:*

$$\begin{bmatrix} V_1 \\ Y \\ U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

where V_1 is the tangent of γ, U is the unit normal vector perpendicular to the tangent plane $T\mathfrak{M}$ of the submanifold $\mathfrak{M}$, and $Y = U \times V_1$. The components

of the Darboux frame of the BCV-Legendre curve γ are given by

$$\begin{bmatrix} \nabla_{V_1} V_1 \\ \nabla_{V_1} Y \\ \nabla_{V_1} U \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ Y \\ U \end{bmatrix}, \tag{2}$$

where $\kappa_n = -\kappa \sin\theta$, $\kappa_g = \kappa \cos\theta$ and $\tau_g = \theta' + \frac{\lambda}{2}$ are defined as the normal curvature, geodesic curvature, and geodesic torsion of the BCV-Legendre curve γ, respectively. Also, θ denotes the angle between the normal vector U and the binormal vector V_3.

Proof. There exists a single angle θ such that a rotation in the plane spanned by V_2 and V_3 produces the pair of vectors Y and U. Hence, the following relations hold:

$$\begin{aligned} g_{\lambda,\mu}(Y, V_2) &= \cos\theta, g_{\lambda,\mu}(Y, V_3) = \sin\theta, \\ g_{\lambda,\mu}(U, V_2) &= -\sin\theta, g_{\lambda,\mu}(U, V_3) = \cos\theta, \end{aligned}$$

which implies that

$$\begin{aligned} Y &= \cos\theta V_2 + \sin\theta V_3, \\ U &= -\sin\theta V_2 + \cos\theta V_3. \end{aligned}$$

Differentiating with respect to s, we obtain

$$\begin{aligned} \nabla_{V_1} Y &= -\kappa\cos\theta V_1 - \sin\theta(\theta' + \frac{\lambda}{2})V_2 + \cos\theta(\theta' + \frac{\lambda}{2})V_3, \\ \nabla_{V_1} U &= \kappa\sin\theta V_1 - \cos\theta(\theta' + \frac{\lambda}{2})V_2 - \sin\theta(\theta' + \frac{\lambda}{2})V_3. \end{aligned}$$

To analyze the relation between the Darboux frame and the Frenet frame, we first express

$$\nabla_{V_1} V_1 = \kappa V_2 = a_1 V_1 + a_2 Y + a_3 U, \tag{3}$$

where

$$\begin{aligned} a_1 &= g_{\lambda,\mu}(\nabla_{V_1} V_1, V_1) = 0, \\ a_2 &= g_{\lambda,\mu}(\nabla_{V_1} V_1, Y) = \kappa\cos\theta = \kappa_g, \\ a_3 &= g_{\lambda,\mu}(\nabla_{V_1} V_1, U) = -\kappa\sin\theta = \kappa_n, \end{aligned}$$

which gives $\nabla_{V_1} V_1 = \kappa_g Y + \kappa_n U$. Taking the norm on both sides of (3), we obtain

$$\kappa = \sqrt{\kappa_g^2 + \kappa_n^2}.$$

Secondly, we can write

$$\nabla_{V_1} Y = b_1 V_1 + b_2 Y + b_3 U,$$

where

$$\begin{aligned} b_1 &= g_{\lambda,\mu}(\nabla_{V_1} Y, V_1) = -\kappa \cos\theta = -\kappa_g, \\ b_2 &= g_{\lambda,\mu}(\nabla_{V_1} Y, Y) = 0, \\ b_3 &= g_{\lambda,\mu}(\nabla_{V_1} Y, U) = \theta' + \frac{\lambda}{2} = \tau_g, \end{aligned}$$

thus $\nabla_{V_1} Y = -\kappa_g V_1 + \tau_g U$. Finally, we have

$$\nabla_{V_1} U = c_1 V_1 + c_2 Y + c_3 U = -\kappa_n V_1 - \tau_g Y,$$

where

$$\begin{aligned} c_1 &= g_{\lambda,\mu}(\nabla_{V_1} U, V_1) = \kappa \sin\theta = -\kappa_n, \\ c_2 &= g_{\lambda,\mu}(\nabla_{V_1} U, Y) = -\theta' - \frac{\lambda}{2} = -\tau_g, \\ c_3 &= g_{\lambda,\mu}(\nabla_{V_1} U, U) = 0, \end{aligned}$$

and consequently, $\nabla_{V_1} U = -\kappa_n V_1 - \tau_g Y$. Thus, the proof is complete; namely, the Darboux frame for the BCV-Legendre curve γ is given by equation (2). □

Remark 3.2. *The Darboux frame $\{V_1, Y, U\}$ provides a geometrically adapted orthonormal basis along the BCV-Legendre curve γ, where each vector has a specific interpretation with respect to the submanifold $\mathfrak{M}$. In particular, while V_1 is the tangent to the curve, U serves as the unit normal vector to the surface $\mathfrak{M}$ at the point $\gamma(s)$, and $Y = U \times V_1$ completes the right-handed frame. This decomposition not only reveals how the curve γ bends and twists within the ambient BCV-Sasakian geometry but also allows the intrinsic and extrinsic geometry of the submanifold to be explicitly expressed via the geodesic curvature κ_g, the normal curvature κ_n, and the geodesic torsion τ_g. These quantities encapsulate the deviation of the curve from being geodesic, its bending relative to the surface $\mathfrak{M}$, and its twisting around the normal vector U, respectively. Notably, since τ_g contains both the derivative of the angle θ and the structural parameter λ of the BCV-space, it reflects the interplay between the geometry of the curve and the underlying contact structure. This frame thus serves as a powerful tool in the geometric analysis of Legendre curves in contact metric manifolds, especially in highlighting the influence of the Sasakian structure on the differential behavior of the curve.*

3.2. Canal Surfaces

A canal surface is defined as the envelope of a one-parameter family of spheres. Let the center curve of the spheres be the BCV-Legendre curve γ with s parameter and Darboux frame on a submanifold $\mathfrak{M}$ in the BCV-Sasakian space. Accordingly, the canal surface $C(s, \phi)$ satisfies the condition

$$\|C(s, \phi) - \gamma(s)\| = r(s), \tag{4}$$

for every point $p = C(s,\phi)$ on the surface, where $r(s)$ denotes the radius function. In this case, the position vector can be expressed in terms of the Darboux frame as

$$C(s,\phi) - \gamma(s) = a(s,\phi)V_1 + b(s,\phi)Y + c(s,\phi)U,$$

where $a(s,\phi)$, $b(s,\phi)$ and $c(s,\phi)$ are smooth functions. Since the canal surface and the family of spheres are tangent along the characteristic circles, their tangent planes coincide at these points. Moreover, the position vector of the sphere coincides with the normal U of the manifold $\mathfrak{M}$ on which the curve lies; therefore, the normal vector of the canal surface can be expressed as

$$U(s,\phi) = C(s,\phi) - \gamma(s).$$

Therefore, the vector $C(s,\phi)-\gamma(s)$ is orthogonal to the tangent vector C_s, i.e.,

$$g_{\lambda,\mu}(C(s,\phi) - \gamma(s), C_s) = 0, \tag{5}$$

where

$$C_s = (1 + a_s - b\kappa_g - c\kappa_n)V_1 + (a\kappa_g + b_s - c\tau_g)Y + (a\kappa_n + b\tau_g + c_s)U.$$

From equations (4) and (5), we have

$$\begin{aligned} a^2 + b^2 + c^2 &= r^2, \\ a + aa_s + bb_s + cc_s &= 0 \end{aligned} \tag{6}$$

and so we reach

$$\begin{aligned} aa_s + bb_s + cc_s &= rr', \\ a &= -rr'. \end{aligned} \tag{7}$$

Finally, from (6) and (7), we obtain

$$\begin{aligned} b &= \pm r\sqrt{1-(r')^2}\cos\phi, \\ c &= \pm r\sqrt{1-(r')^2}\sin\phi, \end{aligned}$$

and the Darboux-Framed BCV-Legendre canal surface is obtained as

$$C(s,\phi) = \gamma(s) - rr'V_1 \pm r\sqrt{1-(r')^2}(\cos\phi Y + \sin\phi U).$$

3.3. Tubes

Let $\gamma(s)$ be a center BCV-Legendre curve lying on the submanifold $\mathfrak{M}$ in the BCV-Sasakian space. Since the characteristic circles of the canal surface lie in planes perpendicular to the tangent vector of $\gamma(s)$, the BCV-Legendre tube can be expressed in terms of the Darboux frame as

$$L(s,\phi) = \gamma(s) + r(\cos\phi Y(s) + \sin\phi U(s)),$$

where U is the unit normal perpendicular to the tangent plane $T_{\gamma(s)}\mathfrak{M}$ of the $\mathfrak{M}$ along the curve $\gamma(s)$. Then, using the derivative formulas of the Darboux

frame, the following expressions are obtained:

$$\begin{aligned} L_s &= (1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi)V_1 - r\tau_g \sin\phi Y + r\tau_g \cos\phi U, \\ L_\phi &= -r\sin\phi Y + r\cos\phi U, \\ L_s \times L_\phi &= -r(1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi)(\cos\phi Y + \sin\phi U) \\ &\quad \|L_s \times L_\phi\|^2 = r^2(1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi)^2. \end{aligned}$$

Thus, the unit normal vector N_L and the first fundamental form coefficients of the BCV-Legendre tube $L(s,\phi)$, expressed in terms of the Darboux frame, are given by:

$$\begin{aligned} N_L &= \frac{L_s \times L_\phi}{\|L_s \times L_\phi\|} = -\cos\phi Y - \sin\phi U, \\ E &= g_{\lambda,\mu}(L_s, L_s) = (1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi)^2 + r^2\tau_g^2, \\ F &= g_{\lambda,\mu}(L_s, L_\theta) = r^2\tau_g, \\ G &= g_{\lambda,\mu}(L_\theta, L_\theta) = r^2. \end{aligned}$$

Then, the first fundamental form of the BCV-Legendre tube $L(s,\phi)$, expressed in the Darboux frame in $\mathfrak{M}^3$, is obtained. Subsequently, by applying the derivative formulas of the Darboux frame, the following expressions are derived:

$$\begin{aligned} L_{ss} &= r(-\kappa_g' \cos\phi - \kappa_n' \sin\phi + \kappa_g\tau_g \sin\phi - \kappa_n\tau_g \cos\phi)V_1 + \\ &\quad (\kappa_g - r\kappa_g^2 \cos\phi - r\kappa_g\kappa_n \sin\phi - r\tau_g' \sin\phi - r\tau_g^2 \cos\phi)Y + \\ &\quad (\kappa_n - r\kappa_n^2 \sin\phi - r\kappa_g\kappa_n \cos\phi + r\tau_g' \cos\phi - r\tau_g^2 \sin\phi)U, \\ L_{\phi s} &= r(\kappa_g \sin\phi - \kappa_n \cos\phi)V_1 - r\tau_g \cos\phi Y - r\tau_g \sin\phi U, \\ L_{\phi\phi} &= -r\cos\phi Y - r\sin\phi U. \end{aligned}$$

As a result, the coefficients of the second fundamental form are calculated as follows:

$$\begin{aligned} e &= g_{\lambda,\mu}(N_L, L_{ss}) = (\kappa_g \cos\phi + \kappa_n \sin\phi)[r(\kappa_g \cos\phi + \kappa_n \sin\phi) - 1] + r\tau_g^2, \\ f &= g_{\lambda,\mu}(N_L, L_{\phi s}) = r\tau_g, \\ g &= g_{\lambda,\mu}(N_L, L_{\phi\phi}) = r. \end{aligned}$$

Consequently, the second fundamental form of the BCV-Legendre tube $L(s,\phi)$ with the Darboux frame in the BCV-Sasakian space is obtained. Moreover, the Gaussian and mean curvatures of the BCV-Legendre surface $L(s,\phi)$ with the Darboux frame are calculated as:

$$K = \frac{\kappa_g \cos\phi + \kappa_n \sin\phi}{r(1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi)}, \tag{8}$$

$$H = \frac{2r(\kappa_g \cos\phi + \kappa_n \sin\phi) - 1}{2r(r\kappa_g \cos\phi + r\kappa_n \sin\phi - 1)}, \tag{9}$$

where the classical expressions are given by $K = \frac{eg-f^2}{EG-F^2}$, $H = \frac{eG-2fF+gE}{2(EG-F^2)}$, and κ_g, κ_n, and τ_g denote the geodesic curvature, normal curvature, and geodesic torsion in the BCV-Sasakian space, respectively.

Corollary 3.3. *Since a surface is said to be regular when $EG - F^2 \neq 0$, the Darboux framed BCV-Legendre tube is regular in the BCV-Sasakian space provided that*

$$r \neq \frac{1}{\kappa_g \cos\phi + \kappa_n \sin\phi}.$$

Example 3.4. *Let $\gamma : I \to \mathfrak{M}^3$ be a Legendre curve in $\mathfrak{M}^3$, given by the parametrization*

$$\gamma(s) = (a\cos s, a\sin s, \frac{\lambda a^2}{2(1+\mu a^2)}s),$$

where a, λ, and μ are constants. The velocity vector field of γ is computed as

$$\dot{\gamma}(s) = -e_1 \sin s + e_2 \cos s,$$

for the specific values $a = 2$, $\lambda = 4$, and $\mu = \frac{1}{4}$. It is also easily verified that $\eta(\dot{\gamma}(s)) = 0$, so $\gamma(s)$ is indeed a Legendre curve. Using the almost contact structure, the Frenet frame vectors along the curve $\gamma(s)$ are obtained as:

$$\begin{aligned} V_1(s) &= \dot{\gamma}(s) = -e_1 \sin s + e_2 \cos s, \\ V_2(s) &= \varphi(V_1(s)) = -e_1 \cos s - e_2 \sin s, \\ V_3(s) &= \xi = e_3. \end{aligned}$$

Then, the components of the Darboux frame $\{V_1, Y, U\}$ along the BCV-Legendre curve γ are obtained as:

$$\begin{aligned} V_1(s) &= -e_1 \sin s + e_2 \cos s, \\ Y(s) &= -e_1 \cos\theta \cos s - e_2 \cos\theta \sin s + e_3 \sin\theta, \\ U(s) &= e_1 \sin\theta \cos s - e_2 \sin\theta \sin s + e_3 \cos\theta. \end{aligned}$$

The Darboux-framed BCV-Legendre tube in $\mathfrak{M}^3$ can be written as:

$$L(s,\phi) = \gamma(s) + r(\cos\phi Y(s) + \sin\phi U(s)),\ 0 \leq \phi \leq 2\pi.$$

Moreover, for $r = \frac{1}{2}$ and

$$L(s,\phi) = (x(s,\phi)\frac{\partial}{\partial x} + y(s,\phi)\frac{\partial}{\partial y} + z(s,\phi)\frac{\partial}{\partial z}),$$

the following expressions are obtained:

$$\begin{aligned} x(s,\phi) &= 2\cos s - \cos s \cos(\phi+\theta), \\ y(s,\phi) &= 2\sin s - \cos s \cos(\phi-\theta), \\ z(s,\phi) &= 4s + \sin(2s)\cos(\phi+\theta) - 2\cos^2 s \cos(\phi-\theta) + \frac{1}{2}\sin(\phi+\theta). \end{aligned}$$

The tubular surface around the BCV-Legendre curve is illustrated in the following figure for $\theta(s) = s$.

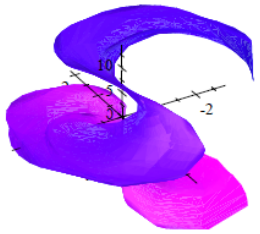

FIGURE 1. The Darboux-framed BCV-Legendre tube

3.4. Some Special Curves

In this section, we examine the relationship between the parameter curves and certain special curves—namely, geodesics, asymptotic curves, and lines of curvature—on the Darboux-framed BCV-Legendre tube.

Theorem 3.5. *For the Darboux-framed BCV-Legendre tube in the BCV-Sasakian space: 1) The ϕ- parameter curves are also geodesics. 2) The s-parameter curves are also geodesics if and only if κ_g, κ_n and τ_g of the center BCV-Legendre curve $\gamma(s)$ satisfy the following system of equations:*

$$(10) \qquad \begin{cases} (\kappa_g \sin\phi - \kappa_n \cos\phi)(1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi) - r\tau_g' = 0 \\ \cos\phi(\kappa_g' + \kappa_n\tau_g) + \sin\phi(\kappa_n' - \kappa_g\tau_g) = 0 \end{cases}$$

Proof. 1)The ϕ-parameter curves are also geodesics curves if and only if

$$N_L \times L_{\phi\phi} = 0.$$

Since $N_L \times L_{\phi\phi} = 0$, it follows that the ϕ- parameter curves are indeed geodesics.

2) The s-parameter curves are also geodesics if and only if

$$N_L \times L_{ss} = 0.$$

Since V_1, Y and U are linearly independent, the condition

$$\begin{aligned} N_L \times L_{ss} \quad = \quad & [(\kappa_g \sin\phi - \kappa_n \cos\phi)(1 - r\kappa_g \cos\phi - r\kappa_n \sin\phi) - r\tau_g']V_{1+} \\ & r(\kappa_g' \cos\phi \sin\phi + \kappa_n' \sin^2\phi - \kappa_g\tau_g \sin^2\phi + \kappa_n\tau_g \cos\phi \sin\phi)Y + \\ & r(-\kappa_g' \cos^2\phi - \kappa_n' \cos\phi \sin\phi + \kappa_g\tau_g \cos\phi \sin\phi - \kappa_n\tau_g \cos^2\phi)U, \end{aligned}$$

implies that all three coefficients must vanish. Therefore, the s-parameter curves are geodesics if and only if the curvature functions κ_g, κ_n and τ_g of the center Legendre curve $\gamma(s)$ satisfy the system of equations given in (10). □

Corollary 3.6. *Let the center BCV-Legendre curve $\gamma(s)$ be a geodesic on $\mathfrak{M}$ in BCV-Sasakian space. If the s-parameter curves are also geodesics on the*

Darboux-framed BCV-Legendre tube, then the curvature of the BCV-Legendre curve $\gamma(s)$ must satisfy the following condition:

$$\kappa' \sin\phi(1 - r\kappa \sin\phi) = 0. \tag{11}$$

Proof. Since the center BCV-Legendre curve $\gamma(s)$ is a geodesic, it follows that $\kappa_g = 0$. Substituting this into the system of equations given in (10), yields the condition presented in equation (11). □

Corollary 3.7. *Let the center BCV-Legendre curve $\gamma(s)$ be an asymptotic curve on $\mathfrak{M}$ in the BCV-Sasakian space. If the s-parameter curves on the Darboux-framed BCV-Legendre tube are also asymptotic curves, then the curvature function κ of $\gamma(s)$ satisfies the following equation:*

$$\kappa' \cos\phi(1 - r\kappa \cos\phi) = 0. \tag{12}$$

Proof. Since the center BCV-Legendre curve $\gamma(s)$ is an asymptotic curve, it follows that $\kappa_n = 0$. Substituting this condition into the system of equations (10) results in equation (12). □

Theorem 3.8. *For the Darboux-framed BCV-Legendre tube in the BCV-Sasakian space, 1) The ϕ- parameter curves cannot be asymptotic curves. 2) The s- parameter curves are asymptotic curves if and only if the Darboux-framed BCV-Legendre tube is generated by a moving sphere whose radius function is given by*

$$r = \frac{\kappa_g \cos\phi + \kappa_n \sin\phi}{(\kappa_g \cos\phi + \kappa_n \sin\phi)^2 + \tau_g^2} = c, \tag{13}$$

where r is constant.

Proof. 1) The ϕ-parameter curves are asymptotic curves if and only if

$$g_{\lambda,\mu}(N_L, L_{\phi\phi}) = 0.$$

Since $g_{\lambda,\mu}(N_L, L_{\phi\phi}) = r$ and $r \neq 0$, the ϕ- parameter curves cannot be asymptotic curves.

2) The s-parameter curves are asymptotic curves if and only if

$$g_{\lambda,\mu}(N_L, L_{ss}) = 0.$$

Then,

$$-(\kappa_g \cos\phi + \kappa_n \sin\phi) + r(\kappa_g \cos\phi + \kappa_n \sin\phi)^2 + r\tau_g^2 = 0,$$

and thus the equality (13) is obtained. □

Corollary 3.9. *Let the s- parameter curves be asymptotic curves on the Darboux-framed BCV-Legendre tube in the BCV-Sasakian space.*

1) If the center BCV-Legendre curve $\gamma(s)$ is a geodesic on $\mathfrak{M}$, then

$$r = \frac{\kappa \sin\phi}{(\kappa \sin\phi)^2 + (\frac{\lambda}{2})^2} = c. \tag{14}$$

2) If the center BCV-Legendre curve $\gamma(s)$ is an asymptotic curve on $\mathfrak{M}$, then

$$r = \frac{\kappa \cos\phi}{(\kappa \cos\phi)^2 + (\frac{\lambda}{2})^2} = c. \tag{15}$$

3) If the center BCV-Legendre curve $\gamma(s)$ is a line on $\mathfrak{M}$, then

$$r = \frac{1}{(\kappa_g \cos\phi + \kappa_n \sin\phi)} = c. \tag{16}$$

Proof. 1) Since the center BCV-Legendre curve $\gamma(s)$ is a geodesic, we have $\kappa_g = 0$. Substituting this into equation (13), yields equation (14).

2) Since the center BCV-Legendre curve $\gamma(s)$ is an asymptotic curve, it follows that $\kappa_n = 0$. Replacement of this in equation (13), results in equation (15).

3) Since the center BCV-Legendre curve $\gamma(s)$ is a line, we have $\tau_g = 0$. Substituting this into equation (13), gives equation (16). □

Theorem 3.10. *The parameter curves of the Darboux-framed BCV-Legendre tube in the BCV-Sasakian space are also lines of curvature if and only if the center BCV-Legendre curve $\gamma(s)$ is itself a line of curvature on $\mathfrak{M}$.*

Proof. According to the classical theorem on lines of curvature, the parameter curves on a surface are also lines of curvature if and only if $F = f = 0$. Since $F = r^2\tau_g$ and $f = r\tau_g$, it follows that $\tau_g = 0$, which implies that the center BCV-Legendre curve $\gamma(s)$ is a line of curvature on $\mathfrak{M}$. □

Theorem 3.11. *For the regular Darboux-framed BCV-Legendre tube in the BCV-Sasakian space, the following results hold:*
1) If the center BCV-Legendre curve $\gamma(s)$ is a geodesic on $\mathfrak{M}$, then the Gaussian curvature K and mean curvature H of the Darboux-framed BCV-Legendre tube are given by:

$$K = \frac{\kappa \sin\phi}{r(1 - r\kappa \sin\phi)}, \tag{17}$$

$$H = \frac{2r(\kappa \sin\phi) - 1}{2r(r\kappa \sin\phi - 1)}, \tag{18}$$

respectively.

2) If the center BCV-Legendre curve $\gamma(s)$ is an asymptotic curve on $\mathfrak{M}$, then the Gaussian curvature K and mean curvature H of the Darboux-framed BCV-Legendre tube are given by:

$$K = \frac{\kappa \cos\phi}{r(1 - r\kappa \cos\phi)}, \tag{19}$$

$$H = \frac{2r(\kappa \cos\phi) - 1}{2r(r\kappa \cos\phi - 1)}, \tag{20}$$

respectively.

Proof. 1) Since the center BCV-Legendre curve $\gamma(s)$ is a geodesic, we have $\kappa_g = 0$. Substituting this into equations (8) and (9), the Gaussian and mean curvatures of the Darboux-framed BCV-Legendre tube are obtained as equations (17) and (18), respectively.

2) Since the center BCV-Legendre curve $\gamma(s)$ is an asymptotic curve, we have $\kappa_n = 0$. Substituting this into equations (8) and (9), the Gaussian and mean curvatures of the Darboux-framed BCV-Legendre tube are obtained as equations (19) and (20), respectively. □

4. Conclusion

In this study, we defined the Darboux frame components for Legendre curves in the BCV Sasakian space. We examined canal and tubular surfaces around the Darboux-framed BCV-Legendre curve, providing new insights into these special geometric structures. In particular, a detailed analysis has been conducted on the Gaussian and mean curvatures of the tubular surfaces, as well as on the relationship of parameter curves with geodesics, asymptotic curves, and lines of curvature, providing comprehensive characterizations that have not been previously addressed in the BCV-Sasakian context.

The originality of this work lies in applying the Darboux frame method in the BCV-Sasakian geometry. The provided example helps to illustrate the theoretical results concretely.

We expect these findings to encourage further research on geometric structures and flows in Sasakian manifolds. Future work could explore the effects of parametric values, different Sasakian structures, and their applications.

5. Acknowledgements

This study was prepared within the scope of the research project titled "Investigation of Some Special Curves and Tube Surfaces in the BCV-Sasakian Manifold and a Preliminary Research on Geometric Pluripotential Theory" supported by TUBITAK (The Scientific and Technological Research Council of Türkiye).

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