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# Special Curves and Tubes in the BCV-Sasakian Manifold

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**Abstract**

In this study, theorems and proofs related to spherical and focal curves are presented in the BCV-Sasakian space. An approximate solution to the differential equation characterizing spherical curves in the BCV-Sasakian manifold $\mathfrak{M}^3$ is obtained using the Taylor matrix collocation method. The general equations of canal and tubular surfaces are provided within this geometric framework. Additionally, the curvature properties of the tubular surface constructed around a non-vertex focal curve are computed and analyzed. All of these results are presented for the first time in the literature within the context of the BCV-Sasakian geometry. Thus, this study makes a substantial contribution to the differential geometry of contact metric manifolds by extending classical concepts into a more generalized and complex geometric structure.

**Keywords:** focal curves; BCV-Sasakian manifold; spherical curves; tubes; Bianchi–Cartan–Vranceanu spaces

## 1. Introduction

While the geometric structure of even-dimensional spaces can be studied using symplectic and complex manifolds, the structure of odd-dimensional spaces can be studied through contact manifolds. Sasakian manifolds, which are odd-dimensional manifolds, are also called normal contact metric manifolds. Legendre curves play an important role in the study of contact manifolds. In contact manifolds, a diffeomorphism is a contact transformation if and only if all Legendre curves in its domain are mapped to Legendre curves [1,2]. Belkhelfa et al. have investigated Legendre curves in Riemannian and Lorentzian Sasaki spaces [3]. The Bianchi–Cartan–Vranceanu (BCV) metric, introduced as a family of Riemannian metrics, has a Sasakian structure on the $E^3$ cartesian space and this structure is defined as the three-dimensional BCV-Sasakian manifold. Legendre curves were studied by Yıldırım and Hacısalihoğlu, who constructed the Frenet 3-frame in the three-dimensional the BCV-Sasakian manifold $\mathfrak{M}^3$ [4].

The theory of curves, a significant field in differential geometry, finds applications in many areas of engineering, such as computer-aided geometric design and manufacturing, robotics, and mechanics. One of the key factors in characterizing a curve is the motion of its associated frame. Special curves are typically defined using a moving frame, and their geometric properties are analyzed accordingly. A curve that lies entirely on a sphere is called a spherical curve, while the curve connecting the centers of the osculating spheres of a curve is called a focal curve. A global formulation for a general curve lying on a sphere and an explicit characterization of the spherical curves in $E^3$ was provided by Wong [5,6]. In

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elementary differential geometry, a necessary and sufficient condition for a curve in $E^3$ to be spherical is that its curvature ($\varkappa$) and torsion ($\tau$) satisfy the following differential equation:

$$\tau\varkappa^{-1} + (\tau^{-1}(\varkappa^{-1})')' = 0.$$

In this expression, the precondition that curvature and torsion cannot be zero anywhere is obvious, and it represents a second-order homogeneous linear differential equation with variable coefficients. It is quite difficult to find analytical solutions to a second- or higher-order homogeneous linear differential equation with variable coefficients. Such equations are typically transformed into normal-form differential equations using developed approximate solution methods, and the solutions are investigated [7].

Ruled surfaces are a class of surfaces generated by a space curve and its associated Frenet frame elements. A canal surface is defined as the envelope of a moving sphere with a variable radius. Canal surfaces, which constitute a special subclass of ruled surfaces, have been the subject of numerous studies. In addition to analyzing the analytical and geometric properties of canal surfaces, singular points of tubular surfaces-which are a special case of canal surfaces-have also been examined [8,9]. When the radius function of the moving sphere that forms the canal surface is constant, the canal surface is called a tube or a pipe surface. The curve that is the orbit of the centers of the moving sphere is known as the center (spine) curve of the tube surface. Tube surfaces are frequently used to model the surfaces of three-dimensional tubular structures commonly observed in real-world applications. They are widely applied in the modeling of solids and surfaces in computer-aided geometric design and manufacturing [10]. Furthermore, tube surfaces have been investigated in various differential geometric frameworks [11].

In this study, spherical and focal curves in the BCV-Sasakian space are investigated for the first time. In addition, this study presents for the first time an approximate solution to the differential equation characterizing spherical curves in the three-dimensional BCV-Sasakian manifold $\mathfrak{M}^3$, obtained using the Taylor matrix collocation method. Although this method was developed in our earlier work and is not newly proposed here, the original contribution of this paper lies in the novel application of this method to a differential equation arising specifically in the BCV-Sasakian space. The obtained solutions reveal the relationship between the radius of curvature and the torsion of the curve. Additionally, the parameterization of a tubular surface constructed around a non-vertex focal curve is presented. For this surface, the surface normal vector, Gaussian and mean curvatures, and the coefficients of the first and second fundamental forms are computed. This article provides new insights into how classical geometric structures behave in more generalized contact metric manifolds and establishes a solid foundation for future research in this area. Furthermore, tubular surfaces exhibit intrinsic geometric symmetries arising from both the structural properties of the BCV-Sasakian space and the characteristics of the focal curves. Sasakian manifolds, equipped with a contact metric structure, induce natural geometric symmetries, while the BCV spaces, being homogeneous, admit a rich family of isometries. When the spine curve possesses regular geometric features such as constant curvature or torsion, the resulting tubular surface often displays rotational or axial symmetry. These symmetries significantly influence the geometric behavior of the surface, including its curvature distribution and the evolution of the Frenet frame. Thus, the study contributes to understanding how the symmetries of the manifold and the generating curve govern the differential geometric properties of the associated tubular surfaces.

## 2. Materials and Methods

### *2.1. The Sasakian Manifold*

A $(2n+1)$-dimensional differentiable manifold $M$ is called a contact manifold if it admits a global 1-form $\eta$ satisfying

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere in $M$. In this case, the 1-form $\eta$ is referred to as a contact structure [1–3]. It is well known that there exists a unique vector field $\xi$ satisfying $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$. This vector field $\xi$ is called the characteristic vector field (or Reeb) associated with $\eta$. The contact distribution $D$ associated with $\eta$ is defined as

$$D = \{X \epsilon \chi(M) : \eta(X) = 0\}.$$

If the $\varphi$, $\eta$ and $\xi$ tensors of type $(1,1)$, $(0,1)$ and $(1,0)$, respectively, satisfy the conditions

$$\begin{aligned} \varphi^2 &= -I + \eta\xi \\ \xi\varphi &= 0 \\ rank\varphi &= 2n \\ \eta(\varphi) &= 0 \end{aligned}$$

then $(\eta, \xi, \varphi)$ is called an almost contact manifold on $M$. In addition, if there exists a Riemannian metric $g$ satisfying

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y) \\ & \quad g(\xi, X) = \eta(X) \end{aligned}$$

on an almost contact structure $(M, \eta, \xi, \varphi)$ then the structure $(M, \eta, \xi, \varphi, g)$ is called an almost contact metric manifold. Furthermore, if the metric $g$ satisfies

$$g(\varphi X, Y) = -d\eta(X, Y)$$

then the structure $(M, \eta, \xi, \varphi, g)$ is called a contact metric manifold. It is well known that every contact metric manifold is a contact manifold [1]. An almost contact metric manifold $(M, \eta, \xi, \varphi, g)$ is a Sasakian manifold if and only if the Levi-Civita connection $\nabla$ satisfies

$$(\nabla_{X\varphi})Y = g(X, Y)\xi - \eta(Y)X,$$

for all vector fields $X, Y$ on $M$. Moreover, if the tensor

$$N^{(1)} : \chi(M) \times \chi(M) \to \chi(M)$$

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi$$

on the Sasakian manifold $(M, \eta, \xi, \varphi, g)$ vanishes then the tensor $N^{(1)}$ is called the Sasakian tensor and the contact manifold $(M, \eta, \xi, \varphi, g)$ is called the Sasakian manifold [1].

*2.2. The BCV-Sasakian Space*

$\{R^3, g_{\lambda,\mu}\}$ is called the BCV space which is denoted by $\mathfrak{M}^3$ or $\mathfrak{M}^3_{\lambda,\mu}$, where $g_{\lambda,\mu}$ is the Bianchi–Cartan–Vranceanu (BCV) metric in $R^3$ and is defined by

$$g_{\lambda,\mu} = \frac{dx_1^2 + dx_2^2}{\left[1+\mu(x_1^2+x_2^2)\right]^2} + [dx_3 + \frac{\lambda}{2}\frac{x_2dx_1 - x_1dx_2}{1+\mu(x_1^2+x_2^2)}]^2 \tag{1}$$

for $\lambda, \mu \in R$ such that $1+\mu(x_1^2+x_2^2) \neq 0$. The dimension of this space is $dim\mathfrak{M}^3 = 3$. If $\lambda = 0$, $\mu = 0$ then $\mathfrak{M}^3$ corresponds to the Euclidean space and denoted by $E^3$. In the special case that $\lambda \neq 0$, $\mu = 0$ the space $\mathfrak{M}^3$ corresponds to the Heisenberg space. Heisenberg space is denoted by $N^3$ [12]. Bianchi classified all Riemannian metrics in the three-dimensional Euclidean space $E^3$ [13]. In the same year, Cartan and in 1947, G. Vranceanu published some papers related to these spaces [14].

Let $\phi = \{e_1, e_2, e_3\}$ be an orthonormal basis of $\chi(\mathfrak{M}^3)$ with respect to the metric (1), where

$$\begin{aligned} e_1 &= \left[1+\mu(x_1^2+x_2^2)\right]\frac{\partial}{\partial x_1} - \frac{\lambda x_2}{2}\frac{\partial}{\partial x_3}, \\ e_2 &= \left[1+\mu(x_1^2+x_2^2)\right]\frac{\partial}{\partial x_2} - \frac{\lambda x_1}{2}\frac{\partial}{\partial x_3}, \\ e_3 &= \frac{\partial}{\partial x_3}. \end{aligned}$$

The dual basis $\theta$ of $\phi$ is given by

$$\begin{aligned} \theta^1 &= \frac{dx_1}{1+\mu(x_1^2+x_2^2)}, \\ \theta^2 &= \frac{dx_2}{1+\mu(x_1^2+x_2^2)}, \\ \theta^3 &= dx_3 + \frac{\lambda}{2}\frac{x_2dx_1 - x_1dx_2}{1+\mu(x_1^2+x_2^2)}. \end{aligned}$$

Furthermore, the covariant derivatives of the basis vectors with respect to each other are given by

$$\begin{bmatrix} \nabla_{e_1}e_1 \\ \nabla_{e_1}e_2 \\ \nabla_{e_1}e_3 \end{bmatrix} = \begin{bmatrix} 0 & 2\mu x_2 & 0 \\ -2\mu x_2 & 0 & \frac{\lambda}{2} \\ 0 & -\frac{\lambda}{2} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$\begin{bmatrix} \nabla_{e_2}e_1 \\ \nabla_{e_2}e_2 \\ \nabla_{e_2}e_3 \end{bmatrix} = \begin{bmatrix} 0 & -2\mu x_1 & -\frac{\lambda}{2} \\ 2\mu x_1 & 0 & 0 \\ \frac{\lambda}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

$$\begin{bmatrix} \nabla_{e_3}e_1 \\ \nabla_{e_3}e_2 \\ \nabla_{e_3}e_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

and

$$\begin{aligned} [e_1, e_2] &= -2\mu x_2 e_1 + 2\mu x_1 e_2 + \lambda e_3, \\ [e_3, e_2] &= 0, \\ [e_1, e_3] &= 0. \end{aligned}$$

Transformation $\varphi$ on $\chi(\mathfrak{M}^3)$, given by

$$\begin{aligned}\varphi(e_1) &= e_2\\ \varphi(e_2) &= -e_1\\ \varphi(e_3) &= 0.\end{aligned}$$

This is a linear endomorphism whose matrix representation is

$$\varphi = \begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}$$

with respect to the basis $\phi$. If $\lambda \neq 0$, $\eta = \theta^3$, and $\xi = e_3$ then the following relations hold

$$\begin{aligned}\varphi(\xi) &= 0,\\ \eta(\xi) &= 1,\\ d\eta(X,Y) &= \frac{\lambda}{2} g_{\lambda,\mu}(X, \varphi(Y)),\\ (\nabla_{X\varphi})Y &= \frac{\lambda}{2}\{g_{\lambda,\mu}(X,Y)\xi - \eta(Y)X\}.\end{aligned} \tag{2}$$

The structure $(\mathfrak{M}^3, \eta, \xi, \varphi, g_{\lambda,\mu})$ satisfying the relations in (2) forms a Sasakian manifold. For $\lambda \neq 0$ it is referred to as a BCV-Sasakian space [3,4,15].

Let $\gamma : I \to \mathfrak{M}^3$ be an arbitrary curve in the BCV-Sasakian space. Let $\{V_1, V_2, V_3\}$ denote the Frenet frame along $\gamma$. When $\eta(\dot{\gamma}) = \delta$ with $0 < \delta < 1$, the Frenet equations are given by

$$\begin{bmatrix} \nabla_{V_1} V_1\\ \nabla_{V_1} V_2\\ \nabla_{V_1} V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} V_1\\ V_2\\ V_3 \end{bmatrix}, \tag{3}$$

where the curvature is $\kappa = \sqrt{\alpha^2 + \beta^2}$, for $\alpha, \beta \in C^\infty(\mathfrak{M}^3, R)$ and the torsion is

$$\tau = \frac{\lambda}{2} + \frac{\alpha\beta' - \beta\alpha'}{\alpha^2 + \beta^2} + \frac{\alpha\delta}{\sqrt{1-\delta^2}}$$

(see [4]).

**Definition 1.** *$(M, \eta, \xi, \varphi, g)$ be a three-dimensional, almost contact metric manifold. The cross product $x$ is defined as*

$$\begin{aligned} XxY &= -g(X, \varphi Y)\xi - \eta(Y)\varphi(X) + \eta(X)\varphi(Y),\\ \|XxY\|^2 &= g(X,X)g(Y,Y) - g(X,Y)^2,\end{aligned}$$

*where $X, Y \in T_P M$ [16].*

### *2.3. The Curves and Surfaces*

Canal surfaces are differential geometric surfaces defined as the envelopes of a one-parameter family of spheres. More precisely, they are generated by a family of spheres whose centers move along a spatial curve $\alpha(t)$, and whose radii are determined by a smooth function $r(t)$. At each parameter value $t$, the sphere $S(t)$ is tangent to the canal surface along a characteristic circle, denoted by $K(t)$. When the radius function is constant, i.e., $r(t) = r$, the resulting surface is commonly referred to in the literature as a tube or cylindrical canal surface.

Accordingly, the canal surface, defined as the envelope of spheres centered at $\alpha(t)$ with radius $r(t)$, admits the following parametric representation:

$$K(t,\theta) = \alpha(t) - r(t)r'(t)\frac{\alpha'(t)}{\|\alpha'(t)\|} \pm r(t)\frac{\sqrt{\|\alpha'(t)\|^2 - r'(t)^2}}{\|\alpha'(t)\|}(\cos\theta N(t) + \sin\theta B(t)),$$

where $N(t)$ and $B(t)$ denote the principal normal and binormal of the curve $\alpha(t)$, respectively. These vectors are the basis vectors of the plane containing the characteristic circle. If the spine curve $\alpha(t)$ is parametrized by arc length (i.e., $\|\alpha'(t)\| = 1$), then the parametrization of the canal surface takes the following simplified form:

$$K(s,\theta) = \alpha(s) - r(s)r'(s)T(s) \pm r(s)\sqrt{1 - r'(s)^2}(\cos\theta N(s) + \sin\theta B(s)).$$

This formulation explicitly reveals the geometric structure of the canal surface in terms of a moving orthonormal frame defined along the spine curve. If the radius function is constant, i.e., $r(s) = r$, the canal surface is known as a tube or pipe surface. In this case, the surface takes the simpler form, as follows:

$$L(s,\theta) = \alpha(s) + r(s)(\cos\theta N(s) + \sin\theta B(s)),\ 0 \le \theta \le 2\pi$$

(see [17,18]).

**Definition 2.** *The set*

$$S_r^{n-1} = \left\{ X = (x_1, x_2, ..., x_n) \in R^n; f(X) = \sum_{i=1}^{n} x_i^2 = r^2, \left|\vec{\nabla} f\right| \neq 0 \right\}$$

*is called a* $(n-1)$*-dimensional hypersphere, where the radius r is constant and* $r \in R$ *[19].*

**Definition 3.** *Let* $\gamma : I \to \mathfrak{M}^3$ *be a curve parametrized by the arc length s. Let the points* $\gamma_m$ *be the centers of the osculating spheres of the curve* $\gamma$*. The curve connecting the points* $\gamma_m$ *is called the focal curve of the curve* $\gamma$ *[17].*

If there are five infinitely close points in the neighborhood of a point with the osculator sphere at $\gamma(s)$ of the curve $\gamma$, it is called a vertex of the curve. Conversely, it is called the non-vertex of the curve. From now on, we assume that all points of the defined curves are non-vertex.

### *2.4. Taylor Matrix Collocation Method*

By the Taylor matrix collocation method, an approximate solution of a high-order variable coefficient, linear differential equation, is obtained in the form of an augmented Taylor series, under certain initial conditions, and in the appropriate range selected. The method is based on the transformation of the differential equation into a matrix equation with the help of collocation points. This matrix equation corresponds to an algebraic equation consisting of unknown Taylor coefficients. Thus, the approximate solution of the differential equation is obtained in the form of a finite Taylor series using the Taylor coefficients found from the solution of the algebraic equation [20,21].

## 3. Results

### *3.1. The Spherical Curves in the BCV-Sasakian Space*

**Definition 4.** *Let* $S_r^2 = \left\{ X \in R^3, g_{\lambda,\mu}(X,X) = r^2 \right\}$ *be defined as a sphere in the BCV-Sasakian space* $\mathfrak{M}^3$*. An arbitrary curve* $\gamma : I \to \mathfrak{M}^3$ *lying on this sphere is called a spherical curve in* $\mathfrak{M}^3$*.*

**Definition 5.** *The sphere that has infinitely close four points in common with the curve $\gamma$ at the point $\gamma(s) \in \gamma$ is called the osculator sphere or curvature sphere at the point $\gamma(s)$ of the curve $\gamma \subset \mathfrak{M}^3$. If the osculator sphere center at point $\gamma(s) \in \mathfrak{M}^3$ is $a$, then it is clear that*

$$\gamma(s) - a = m_1(s)V_1(s) + m_2(s)V_2(s) + m_3(s)V_3(s), \tag{4}$$

*where $m_i$ $(1 \leq i \leq 3)$ is defined as the i-th curvature function of the curve and $\{V_1, V_2, V_3\}$ is the Frenet frame.*

**Theorem 1.** *Let $\gamma : I \to \mathfrak{M}^3$ be a spherical curve with arc length parameter $s$, defined by the coordinate neighborhood $(I, \gamma)$, in the $\mathfrak{M}^3$. The i-th curvature functions of the curve $\gamma$ are*

$$m_1(s) = 0, m_2(s) = -\frac{1}{\kappa(s)}, m_3(s) = -\frac{1}{\tau(s)}(\frac{1}{\kappa(s)})', \tag{5}$$

*where $\kappa(s) = \sqrt{\alpha^2 + \beta^2}$ and $\tau = \frac{\lambda}{2} + \frac{\alpha\beta' - \beta\alpha'}{\alpha^2+\beta^2} + \frac{\alpha\delta}{\sqrt{1-\delta^2}}$.*

**Proof.** Let $\gamma : I \to \mathfrak{M}^3$ be a spherical curve with arc length parameter $s$, defined by the coordinate neighborhood $(I, \gamma)$, in the $\mathfrak{M}^3$. Let the center of the osculator sphere that has four infinite-near points in common with the curve $\gamma$ be $a$ and its radius be $r$ ($a$ and $r$ are constant). Let the function $f : I \to R$ be taken as

$$f(s) = g_{\lambda,\mu}(\gamma(s) - a, \gamma(s) - a) - r^2. \tag{6}$$

If the osculating sphere and the curve $\gamma$ have infinitely close four common points at $\gamma(s)$, then

$$f(s) = f'(s) = f''(s) = f'''(s) = 0.$$

Firstly, it is clear from the equality (4) that $m_1(s) = g_{\lambda,\mu}(\gamma(s) - a, V_1)$, $m_2(s) = g_{\lambda,\mu}(\gamma(s) - a, V_2)$, and $m_3(s) = g_{\lambda,\mu}(\gamma(s) - a, V_3)$. Differentiating (6), then

$$g_{\lambda,\mu}(\gamma(s) - a, V_1) = 0, \tag{7}$$

where $V_1 = \gamma'$, $g_{\lambda,\mu}(V_1, V_1) = 1$. Thus $m_1(s) = 0$. For the case $f''(s) = 0$, the equality (7) is differentiated and

$$g_{\lambda,\mu}(\gamma(s) - a, \kappa V_2) + 1 = 0. \tag{8}$$

Then $m_2(s) = -\frac{1}{\kappa(s)}$, $(\kappa \neq 0)$. Finally, for the case $f'''(s) = 0$, differentiating the (8),

$$g_{\lambda,\mu}(\gamma(s) - a, \tau V_3) + (\frac{1}{\kappa(s)})' = 0, \tag{9}$$

and thus $m_3(s) = -\frac{1}{\tau(s)}(\frac{1}{\kappa(s)})'$, $(\tau \neq 0)$. So, the proof is completed. □

**Corollary 1.** *Let $\gamma : I \to M^3$ be a spherical curve with arc length parameter $s$. The osculator sphere center at point $\gamma(s) \in \gamma$ is*

$$a = \gamma(s) + \frac{1}{\kappa(s)}V_2(s) + \frac{1}{\tau(s)}(\frac{1}{\kappa(s)})'V_3(s). \tag{10}$$

*Also, since $g_{\lambda,\mu}(\gamma(s) - a, \gamma(s) - a) = r^2$, the equality*

$$m_2(s)^2 + m_3(s)^2 = r^2 \tag{11}$$

*is obtained. Then the osculator sphere radius at point* $\gamma(s) \in \gamma$ *is*

$$r = \sqrt{[\frac{1}{\kappa(s)}]^2 + [\frac{1}{\tau(s)}(\frac{1}{\kappa(s)})']^2}. \tag{12}$$

**Theorem 2.** *Let* $\gamma : I \to \mathfrak{M}^3$ *be a curve with the arc length parameter* $s$ *in the BCV-Sasakian space. If* $\gamma$ *is spherical, then there is a correlation between the curvatures defined by*

$$(\frac{1}{\tau(s)})(\frac{1}{\kappa(s)})'' + (\frac{1}{\tau(s)})'(\frac{1}{\kappa(s)})' + \frac{\tau(s)}{\kappa(s)} = 0. \tag{13}$$

**Proof.** Since $r$ is constant, if Equation (12) is differentiated, the proof is completed. Similarly, since $a$ is constant, if Equation (10) is differentiated, the proof can be made using the system (3). □

### *3.2. An Approximate Solution for the Spherical Curves in the BCV-Sasakian Space*

In this section, an approximate solution of the differential equation characterizing spherical curves in $\mathfrak{M}^3$ is given by the Taylor matrix collocation method.

Equation (13), which characterizes spherical curves in $\mathfrak{M}^3$ can be rewritten as follows:

$$\sum_{k=0}^{2} R_k(s)\rho^{(k)}(s) = H(s), \tag{14}$$

where

$$R_0(s) = \tau(s), R_1(s) = (\frac{1}{\tau(s)})', R_2(s) = (\frac{1}{\tau(s)}), \rho(s) = \frac{1}{\kappa(s)}, H(s) = 0.$$

Suppose that Equation (14) has an approximate solution depending on the initial conditions defined as

$$\rho^{(k)}(0) = \rho_k, (k = 0, 1),$$

in the form of a truncated Taylor series, as

$$\rho(s) \cong \rho_N(s) = \sum_{m=0}^{N} a_m s^m, \tag{15}$$

in the range $0 \leq s \leq 2\pi$. The collocation points in the selected range are obtained as

$$s = s_i = \frac{2\pi}{3} i, (i = 0, 1, 2, 3),$$

for $N = 3$. The approximate solution (15) is expressed in matrix form

$$\mathbf{P}(s) = \mathbf{S}(s)\mathbf{A},$$

with $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$ and $\mathbf{S} = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix}$ at these collocation points. The matrices

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $\mathbf{P}'(s) = \mathbf{S}(s)\mathbf{B}\mathbf{A}$ and $\mathbf{P}''(s) = \mathbf{S}(s)\mathbf{B}^2\mathbf{A}$ are evident. Since these equalities will provide the differential Equation (14), we obtain

$$\Big\{\mathbf{R}_2(s)\mathbf{S}(s)\mathbf{B}^2 + \mathbf{R}_1(s)\mathbf{S}(s)\mathbf{B} + \mathbf{R}_0(s)\mathbf{S}(s)\Big\}\mathbf{A} = \mathbf{H}, \tag{16}$$

where

$$\mathbf{R}_k(s) = \begin{bmatrix} R_k(0) & 0 & 0 & 0 \\ 0 & R_k(2\pi/3) & 0 & 0 \\ 0 & 0 & R_k(4\pi/3) & 0 \\ 0 & 0 & 0 & R_k(2\pi) \end{bmatrix}.$$

Additionally, if $\mathbf{R}_2(s)\mathbf{S}(s)\mathbf{B}^2 + \mathbf{R}_1(s)\mathbf{S}(s)\mathbf{B} + \mathbf{R}_0(s)\mathbf{S}(s) = \mathbf{W}$ is taken in Equation (16), this equality turns into

$$\mathbf{W}\mathbf{A} = \mathbf{H} \rightarrow \Big[\ \mathbf{W} \ ; \ \mathbf{H}\ \Big]. \tag{17}$$

The matrix $\mathbf{W}$ is computed and the equation is written as an augmented matrix. Furthermore, the matrix equation of the conditions is obtained as

$$\begin{aligned} \mathbf{P}(0) &= \mathbf{S}(0)\mathbf{A} = \rho_0, \\ \mathbf{P}'(0) &= \mathbf{S}(s)\mathbf{B}\mathbf{A} = \rho_1. \end{aligned}$$

The augmented matrix form of the matrix equation of the conditions is computed as

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & ; & \rho_0 \\ 0 & 1 & 0 & 0 & ; & \rho_1 \end{bmatrix}$$

and

$$\mathbf{U}\mathbf{A} = \mathbf{P} \rightarrow \Big[\ \mathbf{U} \ ; \ \mathbf{P}\ \Big]. \tag{18}$$

It is obvious from Equations (17) and (18) that $\mathbf{W}^*\mathbf{A} = \mathbf{H}^*$ and

$$\Big[\ \mathbf{W}^* \ ; \ \mathbf{H}^*\ \Big] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & ; & 0 \\ w_{10} & w_{11} & w_{12} & w_{13} & ; & 0 \\ 1 & 0 & 0 & 0 & ; & \rho_0 \\ 0 & 1 & 0 & 0 & ; & \rho_1 \end{bmatrix}$$

will be obtained, where

$$\begin{aligned} w_{00} &= \tau(0), w_{01} = (\frac{1}{\tau(0)})', w_{02} = \frac{2}{\tau(0)}, w_{03} = 0, \\ w_{10} &= \tau(\frac{2\pi}{3}), w_{11} = \frac{2\pi}{3}\tau(\frac{2\pi}{3}) + (\frac{1}{\tau(2\pi/3)})', \\ w_{12} &= (\frac{2\pi}{3})^2\tau(\frac{2\pi}{3}) + \frac{4\pi}{3}(\frac{1}{\tau(2\pi/3)})' + \frac{2}{\tau(2\pi/3)}, \\ w_{13} &= (\frac{2\pi}{3})^3\tau(\frac{2\pi}{3}) + \frac{(2\pi)^2}{3}(\frac{1}{\tau(2\pi/3)})' + \frac{4\pi}{\tau(2\pi/3)}. \end{aligned}$$

Thus, the unknown matrix is obtained as $A = \ \mathbf{W}^{*-1} \ \ \mathbf{H}^*\ $, where

$$\begin{aligned} a_0 &= \rho_0, a_1 = \rho_1, \\ a_2 &= \frac{w_{00}w_{13} - w_{03}w_{10}}{w_{12}w_{03} - w_{13}w_{02}}\rho_0 + \frac{w_{11}w_{00} - w_{01}w_{13}}{w_{13}w_{02} - w_{12}w_{03}}\rho_1, \\ a_3 &= \frac{w_{00}w_{12} - w_{02}w_{10}}{w_{13}w_{02} - w_{12}w_{03}}\rho_0 + \frac{w_{01}w_{12} - w_{11}w_{02}}{w_{13}w_{02} - w_{12}w_{03}}\rho_1. \end{aligned}$$

Finally, these expressions are substituted into Equation (15) and the approximate solution is obtained as

$$\begin{aligned}\rho(s) &= \rho_0 + \rho_1 s + \left(\frac{w_{00}w_{13} - w_{03}w_{10}}{w_{12}w_{03} - w_{13}w_{02}}\rho_0 + \frac{w_{11}w_{00} - w_{01}w_{13}}{w_{13}w_{02} - w_{12}w_{03}}\rho_1\right)s^2 + \\ &\left(\frac{w_{00}w_{12} - w_{02}w_{10}}{w_{13}w_{02} - w_{12}w_{03}}\rho_0 + \frac{w_{01}w_{12} - w_{11}w_{02}}{w_{13}w_{02} - w_{12}w_{03}}\rho_1\right)s^3.\end{aligned}$$

### 3.3. Focal Curves in BVC-Sasakian Space

In this section, the focal curves in $\mathfrak{M}^3$ and theorems related to them are presented.

**Lemma 1.** *Let $\gamma : I \to \mathfrak{M}^3$ be a curve parametrized by arc length $s$, and its Frenet frame be $\{V_1, V_2, V_3\}$. Then, the focal curve $\gamma_m$ of $\gamma$ is*

$$\gamma_m = \gamma(s) - m_2 V_2(s) - m_3 V_3(s) \tag{19}$$

*and $-m_2$ and $-m_3$ are the focal coefficients of $\gamma_m$ [17].*

**Theorem 3.** *Let $\gamma : I \to \mathfrak{M}^3$ be a curve parametrized by arc length $s$. Let $\{V_1, V_2, V_3\}$ and $\{v_1, v_2, v_3\}$ be the Frenet frames of the curves $\gamma$ and $\gamma_m$, respectively. Then, we have the following connections:*

$$\begin{aligned} v_1 &= \varkappa V_3 \\ v_2 &= -\varkappa\sigma V_2 \\ v_3 &= \sigma V_1 \end{aligned} \tag{20}$$

*where $\varkappa = \frac{-m_3' - m_2\tau(s)}{|-m_3' - m_2\tau(s)|}$, $\sigma = \frac{\tau(s)}{|\tau(s)|}$, and also $\tau(s)$ is the torsion of the curve $\gamma$.*

**Proof.** Let $s_m$ denote the arc length parameter of the focal curve $\gamma_m$. By differentiating both sides of Equation (19) with respect to the arc length parameter $s$, we reach

$$\frac{d\gamma_m}{ds} = \frac{d\gamma_m}{ds_m}\frac{ds_m}{ds} = (-m_3' - m_2\tau(s))V_3. \tag{21}$$

Taking the norm of both sides of Equation (21) yields

$$\frac{ds}{ds_m} = \frac{1}{|-m_3' - m_2\tau(s)|},$$

and

$$v_1 = \varkappa V_3 = \frac{-m_3' - m_2\tau(s)}{|-m_3' - m_2\tau(s)|}V_3. \tag{22}$$

Differentiating both sides of Equation (22) with respect to the arc length parameter $s$ yields

$$v_2 = -\varkappa\sigma V_2,$$

and

$$\kappa_{\gamma_m} = \frac{|\tau(s)|}{|-m_3' - m_2\tau(s)|}. \tag{23}$$

On the other hand, we may write

$$v_3 = v_1 x v_2 = \sigma V_1,$$

and

$$|\tau_{\gamma_m}| = \frac{|\kappa(s)|}{|-m_3' - m_2\tau(s)|}, \tag{24}$$

where $\kappa_{\gamma_m}$ and $\tau_{\gamma_m}$ are the first and the second curvatures of the curve $\gamma_m$, respectively. □

**Corollary 2.** *Let $\gamma : I \to \mathfrak{M}^3$ be a curve parametrized by the arc length $s$. Since the radius $r$ of its osculating sphere is constant, differentiating Equation (11) with respect to the arc length parameter $s$ yields:*

$$\left(r^2\right)' = 2m_3(m_3' + m_2\tau(s)). \tag{25}$$

*Also, since the curve $\gamma$ is a non-vertex curve, $m_3' + m_2\tau(s) \neq 0$ and thus $m_3 = 0$.*

**Lemma 2.** *Let $\gamma : I \to \mathfrak{M}^3$ be a curve parametrized by the arc length $s$. If its osculating sphere radius $r$ is constant,*

$$r = |m_2| = \frac{1}{\kappa(s)}$$

*and so*

$$\gamma_m(s)' = m_2\tau(s)V_3(s),$$

*where $m_2$ is the first focal coefficient of the focal curve $\gamma_m$.*

**Proof.** Since the curve $\gamma$ is a non-vertex curve, $m_3 = 0$. Using (5), and (25) we find that $r = |m_2| = \frac{1}{\kappa(s)}$ is a constant. Also, by differentiating (4), the equality $\gamma(s)' = m_2\tau(s)V_3(s)$ is reached. □

**Corollary 3.** *If we consider (23)–(25) then we obtain that*

$$\frac{\kappa_{\gamma_m}}{|\tau(s)|} = \frac{|\tau_{\gamma_m}|}{|\kappa(s)|} = \frac{1}{|-m_3' - m_2\tau(s)|} = \frac{2|m_3|}{\left|(r^2)'\right|}.$$

**Lemma 3.** *Let $s_m$ be the arc length parameter of the focal curve $\gamma_m$ and $\{v_1, v_2, v_3\}$ be the Frenet frame of the curve $\gamma_m$. According to the parameter $s$, derivative changes in Frenet frame $\{v_1, v_2, v_3\}$ is*

$$\begin{bmatrix} \nabla_{v_1}v_1 \\ \nabla_{v_1}v_2 \\ \nabla_{v_1}v_3 \end{bmatrix} = \begin{bmatrix} 0 & \vartheta\kappa_{\gamma_m} & 0 \\ -\vartheta\kappa_{\gamma_m} & 0 & \vartheta\tau_{\gamma_m} \\ 0 & -\vartheta\tau_{\gamma_m} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

*where $\vartheta = \frac{ds_m}{ds} = |-m_3' - m_2\tau(s)|$. Also, if the radius $r$ of the osculating sphere is constant, then*

$$\vartheta = r|\tau(s)|. \tag{26}$$

**Example 1.** *Let $\gamma(s) = (r\cos s, r\sin s, c)$ be a regular curve in the BCV-Sasakian space $\mathfrak{M}^3$. The velocity vector of this curve is*

$$\gamma'(s) = (-\frac{r\sin s}{1+\mu r^2}e_1 + \frac{r\cos s}{1+\mu r^2}e_2 - \frac{r^2}{1+\mu r^2}e_3).$$

*Assuming that the Frenet frame of $\gamma$ is $\{V_1, V_2, V_3\}$, we obtain*

$$V_1 = -\frac{\sin s}{\sqrt{1+r^2}}e_1 + \frac{\cos s}{\sqrt{1+r^2}}e_2 - \frac{r}{\sqrt{1+r^2}}e_3$$

*and*

$$\varphi V_1 = -\frac{\cos s}{\sqrt{1+r^2}}e_1 - \frac{\sin s}{\sqrt{1+r^2}}e_2.$$

*Then, using the Levi-Civita connection in* $\mathfrak{M}^3$*, we find*

$$\nabla_{V_1} V_1 = (\frac{1}{r\sqrt{1+r^2}} + \frac{\lambda}{2\sqrt{1+r^2}} - 2\mu r^2)\varphi V_1.$$

*Hence, the functions* $\alpha$ *and* $\beta$ *are computed*

$$\alpha = \frac{2+\lambda r^2}{2r(1+r^2)} - 2\mu r^2, \beta = 0.$$

*This yields the curvature and torsion of the curve* $\gamma$ *as*

$$\kappa = \left| \frac{2+\lambda r^2}{2r(1+r^2)} - 2\mu r^2 \right|, \tau = \frac{\lambda - 2}{2(1+r^2)} + 2\mu r^3.$$

*Also, the focal curve* $\gamma_m$ *of* $\gamma$ *is computed as*

$$\gamma_m = \gamma(s) + \frac{1}{\kappa} V_2.$$

*A graphical illustration of the curve* $\gamma$ *and its focal curve* $\gamma_m$ *is provided in Figure 1 for the parameter values* $r = 2$, $c = 1$, *and* $\kappa = \frac{1}{\sqrt{2}}$.

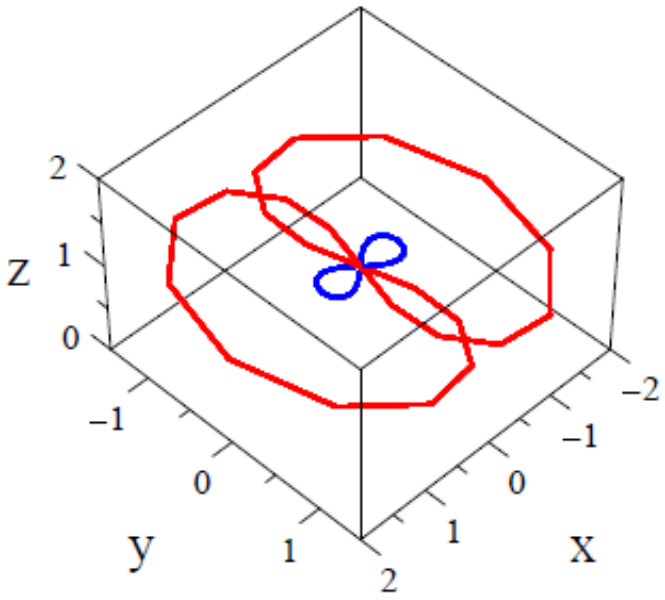


**Figure 1.** The curve $\gamma$ and the focal curve $\gamma_m$ of $\gamma$.

This graphical representation illustrates the spatial relationship between the original curve and its focal counterpart in the BCV-Sasakian space, visually supporting the theoretical derivation of the Frenet apparatus and curvature functions.

### *3.4. Tubular Surfaces in the BCV-Sasakian Space*

**Theorem 4.** *Suppose* $\gamma : I \to \mathfrak{M}^3$ *be a unit speed curve in the BCV-Sasakian space. Then the canal surface around this curve is parametrized as*

$$K(s,\theta) = \gamma(s) - r(s)r'(s)V_1(s) \pm r(s)\sqrt{1-r'(s)^2}(\cos\theta V_2(s) + \sin\theta V_3(s)), \quad (27)$$

*where* $V_1$, $V_2$ *and* $V_3$ *denote the Frenet vector fields of* $\gamma$. *Also, the curvatures* $\kappa$ *and* $\tau$ *of* $\gamma$ *are non-zero.*

**Proof.** Let $K$ denote a patch that parametrizes the envelope of the spheres that define the canal surface. Since the curvature of $\gamma$ is nonzero, the Frenet–Serret frame $\{V_1, V_2, V_3\}$ is well-defined, and we can write

$$K(s,\theta) - \gamma(s) = t(s,\theta)V_1(s) + n(s,\theta)V_2(s) + p(s,\theta)V_3(s), \tag{28}$$

where $t$, $n$ and $p$ are differentiable on the interval on which $\gamma$ is defined. We must have

$$g_{\lambda,\mu}(K(s,\theta) - \gamma(s), K(s,\theta) - \gamma(s)) = r^2. \tag{29}$$

Equation (29) expresses analytically the geometric fact that $K(s,\theta)$ lies on a sphere $S_r^2(s)$ of radius $r(s)$ centered at $\gamma(s)$. Furthermore, $K(s,\theta) - \gamma(s)$ is a normal vector to the canal surface; this fact implies that

$$g_{\lambda,\mu}(K(s,\theta) - \gamma(s), K_s(s,\theta)) = 0 \tag{30}$$

$$g_{\lambda,\mu}(K(s,\theta) - \gamma(s), K_\theta(s,\theta)) = 0. \tag{31}$$

Equations (29) and (30) say that the vectors $K_s$ and $K_\theta$ are tangent to $S_r^2(s)$. From (28) and (29) we obtain

$$\begin{array}{c} t^2 + n^2 + p^2 = r^2 \\ tt_s + nn_s + pp_s = rr'. \end{array} \tag{32}$$

When we differentiate (28) with respect to $s$ and use the Frenet–Serret formulas, we get

$$K_s(s,\theta) = (1 + t_s - n\kappa)V_1 + (t\kappa + n_s - \tau p)V_2 + (n\tau + p_s)V_3. \tag{33}$$

Then (28), (30), (32) and (33) imply that

$$t + rr' = 0, \tag{34}$$

and from (32) and (34) we get

$$n^2 + p^2 = r^2(1 - r'^2). \tag{35}$$

We can write

$$\begin{array}{rcl} n & = & \pm r(s)\sqrt{1 - r'(s)^2}\cos\theta, \\ p & = & \pm r(s)\sqrt{1 - r'(s)^2}\sin\theta. \end{array}$$

The proof is completed by using $t$, $n$ and $p$ in the equality (28). □

**Corollary 4.** *Suppose $\gamma : I \to \mathfrak{M}^3$ is a unit speed curve that is not geodesic in the BCV-Sasakian space. Then the tubular surface around this curve is parametrized as*

$$L(s,\theta) = \gamma(s) + r(s)(\cos\theta V_2(s) + \sin\theta V_3(s)),\ 0 \le \theta \le 2\pi. \tag{36}$$

**Proof.** If $r(s) = r$ is a constant, the canal surface is called a tube or pipe surface, the equality (27) takes the form (36). □

**Theorem 5.** *Let us assume that $\gamma$ be a curve parametrized by arc length $s$ in $\mathfrak{M}^3$, and the curve $\gamma_m$ is the focal curve of $\gamma$. Then the tubular surface around the focal curve $\gamma_m$ is parametrized as*

$$M(s,\theta) = \gamma_m + r(\cos\theta v_2(s) + \sin\theta v_3(s)),\ 0 \le \theta \le 2\pi. \tag{37}$$

**Proof.** If the focal curve $\gamma_m$ and the system (20) are used in the equality (36), the equality (37) is obtained. □

*3.5. The Curvatures of Tubular Surfaces Around a Focal Curve*

For the tubular surface $M(s,\theta)$ around the nonvertex focal curve $\gamma_m$, the normal surface vector $U$ and the coefficients of the first and second fundamental forms are defined by

$$M_\theta = r(-\sin\theta v_2(s) + \cos\theta v_3(s)),$$

$$M_s = \vartheta[(1 - r\kappa_{\gamma_m}\cos\theta)v_1 + \tau_{\gamma_m}M_\theta],$$

$$U = \frac{M_s x M_\theta}{\|M_s x M_\theta\|} = -\cos\theta v_2(s) - \sin\theta v_3(s),$$

$$M_{\theta\theta} = -r(\cos\theta v_2(s) + \sin\theta v_3(s)),$$

$$\begin{aligned} M_{ss} &= \vartheta r(\vartheta\kappa_{\gamma_m}\tau_{\gamma_m}\sin\theta - \kappa'_{\gamma_m}\cos\theta)v_1(s) \\ &\quad +\vartheta[\vartheta\kappa_{\gamma_m} - \vartheta r\cos\theta(\kappa^2_{\gamma_m} + \tau^2_{\gamma_m}) - r\tau'_{\gamma_m}\sin\theta]v_2(s) \\ &\quad +\vartheta r(\tau'_{\gamma_m}\cos\theta - \vartheta\tau^2_{\gamma_m}\sin\theta)v_3(s), \end{aligned}$$

$$M_{s\theta} = \vartheta r[\kappa_{\gamma_m}\sin\theta v_1(s) - \tau_{\gamma_m}\cos\theta v_2(s) - \tau_{\gamma_m}\sin\theta v_3(s)],$$

$$E = g_{\lambda,\mu}(M_s, M_s) = \vartheta^2\left[(1 - r\kappa_{\gamma_m}\cos\theta)^2 + (r\tau_{\gamma_m})^2\right],$$

$$F = g_{\lambda,\mu}(M_s, M_\theta) = \vartheta r^2\tau_{\gamma_m},$$

$$G = g_{\lambda,\mu}(M_\theta, M_\theta) = r^2,$$

$$e = g_{\lambda,\mu}(U, M_{ss}) = \vartheta\left[\kappa_{\gamma_m}\cos\theta(r\kappa_{\gamma_m}\cos\theta - 1) + r\tau^2_{\gamma_m}\right],$$

$$f = g_{\lambda,\mu}(U, M_{s\theta}) = \vartheta r\tau_{\gamma_m},$$

$$g = g_{\lambda,\mu}(U, M_{\theta\theta}) = r,$$

$$\|M_s x M_\theta\|^2 = EG - F^2 = \vartheta^2 r^2(1 - r\kappa_{\gamma_m}\cos\theta)^2. \tag{38}$$

**Theorem 6.** *The tubular surface $M(s,\theta)$ around the non-vertex focal curve $\gamma_m$ in the BCV-Sasakian space is regular if and only if*

$$\cos\theta \neq \frac{1}{r\kappa_{\gamma_m}}.$$

**Proof.** For a regular surface, $EG - F^2 \neq 0$. By Equation (38), since $\vartheta^2 r^2 > 0$, we have $1 - r\kappa_{\gamma_m}\cos\theta$. This completes the proof. □

On the other hand, the Gaussian and mean curvature for a regular tube $M(s,\theta)$ around the non-vertex focal curve $\gamma_m$ are computed as

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} = \frac{r\tau^2_{\gamma_m}(1 - \vartheta) - \kappa_{\gamma_m}\cos\theta(1 - r\kappa_{\gamma_m}\cos\theta)}{(1 - r\kappa_{\gamma_m}\cos\theta)}, \\ H &= \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{r\tau^2_{\gamma_m}(1 - \vartheta) - \kappa_{\gamma_m}\cos\theta(1 - r\kappa_{\gamma_m}\cos\theta)}{2\vartheta(1 - r\kappa_{\gamma_m}\cos\theta)^2}. \end{aligned} \tag{39}$$

**Theorem 7.** *If the Gaussian curvature $K$ is zero for $\vartheta = 1$, then $M(s,\theta)$ is generated by a moving sphere with radius $r = 1$ and $|\tau(s)| = 1$.*

**Proof.** When $K = 0$ for $\vartheta = 1$, from Equation (39) $\cos\theta = 0$ and so the normal of $M(s,\theta)$ becomes $U = \pm v_3(s)$ and

$$M(s,\theta) - \gamma_m = r\sin\theta v_3(s) = \pm r v_3(s).$$

Also

$$U = M(s,\theta) - \gamma_m.$$

From the last equality, we must have $r = 1$. Additionally, for $\vartheta = 1$ and $r = 1$ we obtain the $|\tau(s)| = 1$ from equality (26). □

**Theorem 8.** *Let $M(s,\theta)$ be a regular tube around the non-vertex focal curve $\gamma_m$ in the BCV-Sasakian space. In this case, we have the following:*

*(1) The s parameter curves of $M(s,\theta)$ are also asymptotic curves if and only if*

$$\frac{\tau_{\gamma_m}^2}{\kappa_{\gamma_m}} = \frac{1}{r}\cos\theta(1 - r\kappa_{\gamma_m}\cos\theta). \tag{40}$$

*(2) The θ parameter curves of $M(s,\theta)$ cannot also be asymptotic curves.*

**Proof.** (1) A non-vertex focal curve $\gamma_m$ lying on a surface is an asymptotic curve if and only if the acceleration vector $\gamma_m''$ is tangent to the surface, that is $g_{\lambda,\mu}(U,\gamma_m'') = 0$. Then, for the $s$ parameter curves of $M(s,\theta)$ in the BCV-Sasakian space we have

$$e = g_{\lambda,\mu}(U, M_{ss}) = \vartheta\left[\kappa_{\gamma_m}\cos\theta(r\kappa_{\gamma_m}\cos\theta - 1) + r\tau_{\gamma_m}^2\right] = 0.$$

From this we get the equality (40).

(2) For the $\theta$ parameter curves we have

$$g = g_{\lambda,\mu}(U, M_{\theta\theta}) = r \neq 0,$$

thus $\theta$ parameter curves cannot be asymptotic. □

**Theorem 9.** *Let $M(s,\theta)$ be a regular tube around the non-vertex focal curve $\gamma_m$.*

*(1) The θ parameter curves of $M(s,\theta)$ are also geodesics.*

*(2) The s parameter curves are also geodesics of $M(s,\theta)$ if and only if the curvatures of $\gamma_m$ satisfy the equation*

$$r\kappa_{\gamma_m}^2\cos^2\theta - 2\kappa_{\gamma_m}\cos\theta + r\tau_{\gamma_m}^2 = c$$

*where c is a constant.*

**Proof.** A curve $\gamma_m$ lying on a surface is a geodesic curve if and only if the acceleration vector $\gamma_m''$ is normal to the surface. This means that $\gamma_m''$ and the surface normal $U$ are linearly dependent, namely $Ux\gamma_m'' = 0$. In this case, for the $s$ and $\theta$ parameter curves we conclude

$$UxM_{\theta\theta} = r(\sin\theta\cos\theta v_1 - \sin\theta\cos\theta v_1) = 0,$$

$$\begin{aligned} UxM_{ss} &= \vartheta[-r\tau_{\gamma_m}' + \vartheta\kappa_{\gamma_m}\sin\theta(1 - r\kappa_{\gamma_m}\cos\theta)]v_1(s) \\ &\quad +[-\vartheta r\sin\theta(\vartheta\kappa_{\gamma_m}\tau_{\gamma_m}\sin\theta - \kappa_{\gamma_m}'\cos\theta)]v_2(s) \\ &\quad +\vartheta r\cos\theta(\vartheta\kappa_{\gamma_m}\tau_{\gamma_m}\sin\theta - \kappa_{\gamma_m}'\cos\theta)v_3(s). \end{aligned}$$

(1) As already seen, the $\theta$ parameter curves of $M(s,\theta)$ are also geodesics.

(2) Since $\{v_1, v_2, v_3\}$ is an orthonormal basis, $UxM_{ss}$ if and only if

$$\begin{aligned} -r\tau'_{\gamma_m} + \vartheta\kappa_{\gamma_m}\sin\theta(1 - r\kappa_{\gamma_m}\cos\theta) &= 0 \\ \vartheta r\sin\theta(\vartheta\kappa_{\gamma_m}\tau_{\gamma_m}\sin\theta - \kappa'_{\gamma_m}\cos\theta) &= 0 \\ \vartheta r\cos\theta(\vartheta\kappa_{\gamma_m}\tau_{\gamma_m}\sin\theta - \kappa'_{\gamma_m}\cos\theta) &= 0. \end{aligned} \tag{41}$$

By the last two equations we have $\vartheta\kappa_{\gamma_m}\tau_{\gamma_m}\sin\theta - \kappa'_{\gamma_m}\cos\theta = 0$. If this equation is solved with the first equation of (41) it concludes that

$$\kappa'_{\gamma_m}\cos\theta - r\kappa_{\gamma_m}\kappa'_{\gamma_m}\cos^2\theta - r\tau_{\gamma_m}\tau'_{\gamma_m} = 0.$$

Because $\theta$ is a constant, if we take integral of the above differential equation we obtain

$$r\kappa^2_{\gamma_m}\cos^2\theta - 2\kappa_{\gamma_m}\cos\theta + r\tau^2_{\gamma_m} = c.$$

□

## 4. Conclusions and Discussion

This study presents significant and original contributions to the field of differential geometry within the context of BCV-Sasakian space. Moving beyond the commonly studied Legendre curves, two distinct types of special curves—spherical curves and focal curves—are examined in detail for the first time in this setting. Notably, this study derives a second-order, variable-coefficient, homogeneous linear differential equation characterizing spherical curves in the BCV-Sasakian space and presents an approximate solution using a novel matrix ordering method. This approach represents a completely original and foundational method in the context of differential geometry.

Moreover, the Frenet frame components for focal curves in the BCV-Sasakian space are introduced for the first time in this work. Additionally, the first and second fundamental forms, as well as the Gaussian and mean curvatures of the tube surface generated around a focal curve, are computed. These analyses fill previously unexplored gaps in the literature and provide a solid theoretical foundation for future research. This study is further supported by a concrete example, validating the results obtained.

As summarized in Figure 2, while previous studies often focus separately on spherical or Legendre curves, none address the combination of spherical curves, focal curves, and tubular surfaces constructed around focal curves in the BCV-Sasakian manifold. This study is the first to bring these elements together in a comprehensive manner, highlighting how the manifold's structural symmetries and the geometric properties of the generating curves influence the differential geometry of the resulting tubular surfaces.

| **Author(s) & Year** | **Manifold / Setting** | **Key Results** | **Novelty / Advantages** |
|---|---|---|---|
| Yıldırım (2016) | BCV-Sasakian | Frenet frame-based tubular surface defined | First detailed tubular surfaces in BCV-Sasakian space |
| Blair & Baikoussis (1994) | Contact metric manifold | Studies on Legendre curves | Did not address tubular surfaces in BCV-Sasakian space |
| Inoguchi et al. (1999) | Heisenberg 3-space | Surfaces studied in Heisenberg space | No prior work on BCV-Sasakian tubular surfaces |
| **This Study (2025)** | BCV-Sasakian | Approximate solution for spherical curves; tubular surfaces around focal curves parameterized; curvature properties computed | First comprehensive study of tubular surfaces around focal curves in BCV-Sasakian space |

**Figure 2.** Comparative summary of key studies in the BCV-Sasakian and contact geometry [1,12,17].

Taken together, the article offers a new perspective on curve theory in the BCV-Sasakian geometry and contributes original insights to the field, potentially opening new avenues for exploring other special curves and surfaces in this space.

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## References

1. Baikousis, C.; Blair, D.A. On Legendre curves in contact 3-manifolds. *Geom. Dedicata* **1994**, *49*, 135–142. [CrossRef]
2. Camci, C.; Yayli, Y.; Hacisalihoglu, H.H. On the characterization of spherical curves in three-dimensional Sasakian spaces. *J. Math. Anal. Appl.* **2008**, *342*, 1151–1159. [CrossRef]
3. Belkhelfa, M.; Hırıca, I.E.; Rosca, R.; Verstraelen, L. On Legendre curves in Riemannian and Lorentzian Sasaki spaces. *Soochow J. Math.* **2002**, *28*, 81–91.
4. Yildirim, A.; Hacisalihoglu, H.H. On BCV-Sasakian Manifold. *Int. Math. Forum* **2011**, *6*, 1669–1684.
5. Wong, Y.-C. A global formulation of the condition for a curve to lie in a sphere. *Monatsh. Math.* **1963**, *67*, 363–365. [CrossRef]
6. Wong, Y.-C. On an explicit characterization of spherical curves. *Proc. Am. Math. Soc.* **1972**, *34*, 239–242. [CrossRef]
7. Karamete, A.; Sezer, M. A Taylor collocation method for the solution of linear integro-differential equations. *Int. J. Comput. Math.* **2002**, *79*, 987–1000. [CrossRef]
8. Xu, Z.; Feng, R.; Sun, J. Analytic and algebraic properties of canal surface. *J. Comput. Appl. Math.* **2006**, *195*, 220–228. [CrossRef]
9. Es, H.; Karacan, M.K.; Yaylı Y. Singular points of tubular surface. *J. Xuzhou Norm. Univ. (Nat. Sci. Ed.)* **2009**, *27*, 30–33.
10. Dede, M.; Ekici, C.; Tozak, H. Directional tubular surfaces. *Int. J. Algebra* **2015**, *9*, 527–535. [CrossRef]
11. Yaylı Y.; Doğan, F. Tubes with Darboux Frame. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 751–758.
12. Inoguchi, J.I.; Kumamoto, T.; Ohsugi, N.; Suyama, Y. Differential geometry of curves and surfaces in three-dimensional homogeneous spaces 1. *Fukuoka Univ. Sci. Rep.* **1999**, *29*, 155–182.
13. Bianchi, L. *Lezioni di Geometrie Differenziale*; Enrico Spoerri: Pisa, Italy, 1894; p. 552.
14. Cartan, E. Leçons sur la Géométrie des Espaces de Riemann. In *Lecons sur la Geometria des Espace de Riemann*; G. Villars: Paris, France, 1894.
15. Belkhelfa, M.; Dillen, F.; Inoguchi, J. Surfaces with parallel second fundamental form in Bianchi-Cartan-Vranceanu spaces. *Banach Center Publ.* **2002**, *57*, 67–87.
16. Camcı, Ç. Extended cross product in a three-dimensional almost contact metric manifold with applications to curve theory. *Turk. J. Math.* **2012**, *36*, 305–318.
17. Yildirim, A. Tubular surface around a Legendre curve in BCV spaces. *NTMSCI* **2016**, *4*, 61–71. [CrossRef]
18. Doğan, F.; Yaylı, Y. On the curvatures of tubular surface with Bishop frame. *Commun. Fac. Sci. Univ. Ank. Ser. A1* **2011**, *60*, 59–69.
19. Hacısalihoğlu, H.H. Diferansiyel Geometri-Cilt II. In *Diferansiyel Geometri*; University of Ankara Pub.: Ankara, Türkiye, 2000.
20. Sezer, M. A method for the approximate solution of the second order linear differential equations in terms of Taylor Polynomials. *Int. J. Math. Educ. Sci. Technol.* **1996**, *27*, 821–834. [CrossRef]
21. Agirman Aydin, T.; Sezer, M. Taylor-matrix collocation method to solution of differential equations characterizing spherical curves in Euclidean 4-Space. *Celal Bayar Univ. J. Sci.* **2019**, *15*, 1–7. [CrossRef]

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